# Application of Kelvin's Inversion Theorem to the Solution of Laplace's Equation Over a Domain That Includes the Unbounded Exterior of a Sphere

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## Abstract

In the simulation of physical systems modeled as solutions of partial differential equations over an unbounded domain, a challenge arises immediately owing to the impossibility to cover an unbounded domain with a mesh. The talk addresses this challenge for three-dimensional simulations, taking a classical problem in Fluid Dynamics as an example.

A solid sphere of radius a_s is submerged in a tank of water of much larger size. The water is initially at rest and is disturbed by the sudden acceleration of the sphere to speed Q in the direction of some ray, which I will take to be the positive x_3-, or z-axis. If the acceleration is sudden enough, then the sphere will translate only a short distance compared to its radius during the start-up and the fluid motion just afterwards is irrotational. The fluid velocity vector is then the gradient of a potential, phi. Treating the water as incompressible, the fluid velocity is divergence-free and phi becomes a solution of Laplace's equation. The exact solution for phi, of nineteenth century vintage, is (-1/2)*Q*x_3*(a_s/r)^3, in which r is the radial coordinate denoting distance from the center of the sphere. One may generate this solution in COMSOL Multiphysics® by decomposing the exterior of the sphere into two subdomains, of which one is the exterior of a larger sphere of radius a, which I call the reflecting sphere. Given a position arrow with cartesian coordinates (x_1, x_2, x_3), which I will call the x-arrow, one may define the coordinates (q_1, q_2, q_3) of a corresponding q-arrow that points the same direction as the x-arrow and represents the image of the x-arrow as reflected by the larger sphere by the rule x_i=(a/q)^2*q_i, i∈{1,2,3}, in which q is the magnitude of the q-arrow. In the 1840s, Lord Kelvin proved that if phi is a solution of Laplace's equation in a domain whose coordinates are (x_1, x_2, x_3), then Phi=(a/q)phi is a solution of Laplace's equation in the corresponding image domain with coordinates (q_1, q_2, q_3). The exterior of the reflecting sphere in the x-coordinates maps to an inverted exterior in the q-coordinates -- which looks like an interior -- and is suitable for meshing. COMSOL allows a simultaneous solution for phi in the region where r is between a_s and a (Figure 1) and for Phi in the inverted exterior (Figure 2). In accordance with the predictions of Kelvin's theorem, the results are not sensitive to the ratio a_s/a. Figures 3 and 4 correspond to Figures 1 and 2, respectively, but with the ratio a_s/a increased from 1/3 to 2/3 and there is no mismatch between the COMSOL and analytical solutions in either case.

## Reference

[1] William Thomson, Extrait d'une lettre de M. William Thomson à M. Liouville. Journal de Mathématiques, Vol. 10, 1845. [In Lord Kelvin, Reprint of Papers on Electrostatics and Magnetism Cambridge 1884, pages 144-146]

[2] William Thomson, Extraits de deux lettre adressées à M. Liouville par M. William Thomson. Journal de Mathématiques, Vol. 12, 1847. [In Lord Kelvin, Reprint of Papers on Electrostatics and Magnetism, Cambridge 1884, pages 146-154]

## Figures used in the abstract

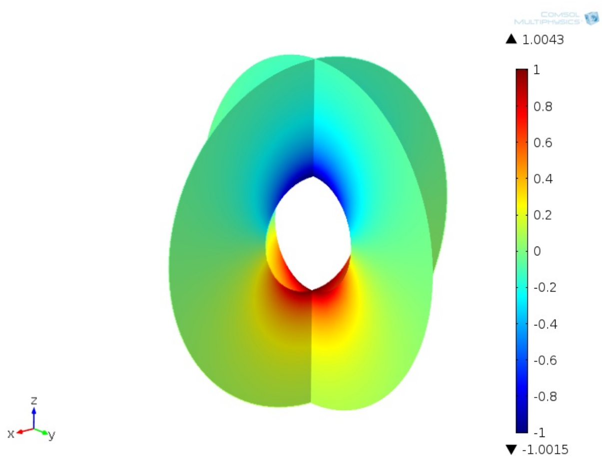


**Figure 1**: Comparison between the COMSOL simulation of phi/phi_scale (the y-z plane) and the analytic solution (the z-x plane). The ratio a_s/a equals 1/3 and phi_scale is the value of phi at z=a_s. The domain is the region between the solid sphere and the reflecting sphere, so the position coordinates are (x_1, x_2, x_3)=(x,y,z). Here, and elsewhere, Q=1[m/s] and a=1[m].

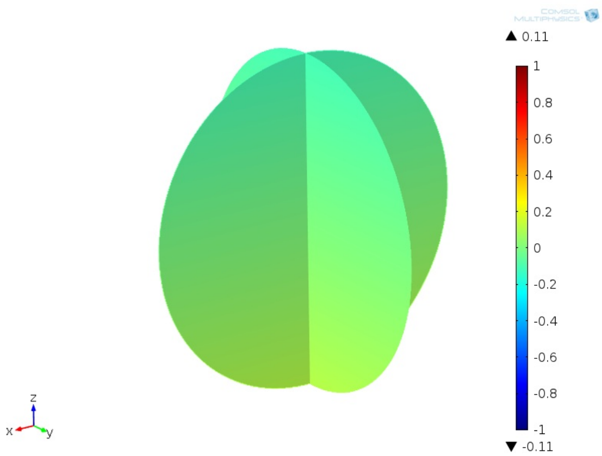


**Figure 2**: Comparison between the COMSOL simulation of Phi/phi_scale (the y-z plane) and the analytic solution (the z-x plane). The domain is the inverted exterior of the reflecting sphere, so the position coordinates are (q_1, q_2, q_3)=(x,y,z). Other parameters are as in Figure 1.

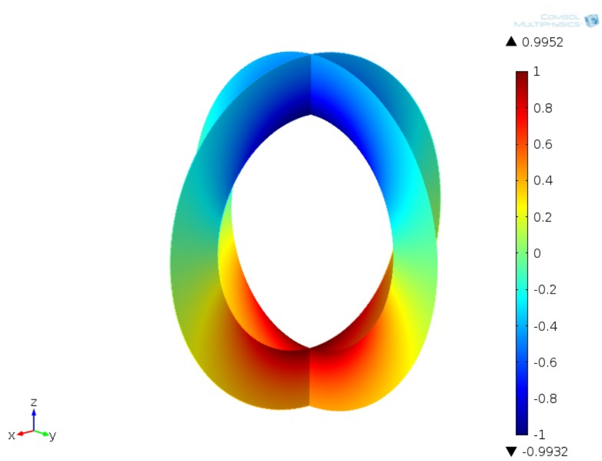


**Figure 3**: Comparison between the COMSOL simulation of phi/phi_scale (the y-z plane) and the analytic solution (the z-x plane). The ratio a_s/a equals 2/3 and phi_scale is the value of phi at z=a_s. The domain is the region between the solid sphere and the reflecting sphere, so the position coordinates are (x_1, x_2, x_3)=(x,y,z). The data replicates those of Figure 1 within a spherical subdomain of radius 1/2[m].

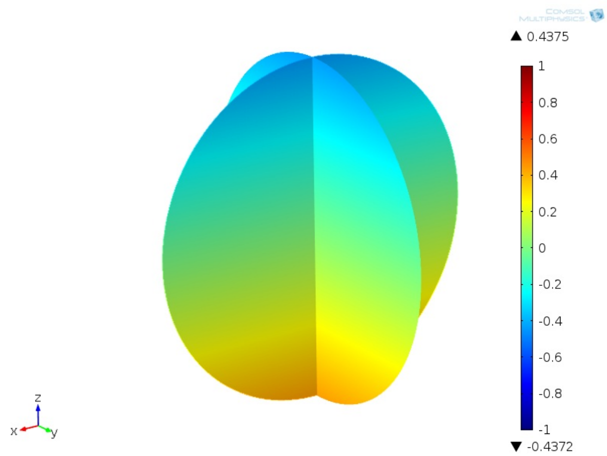


**Figure 4**: Comparison between the COMSOL simulation of Phi/phi_scale (the y-z plane) and the analytic solution (the z-x plane). The domain is the inverted exterior of the reflecting sphere, so the position coordinates are (q_1, q_2, q_3)=(x,y,z). Other parameters are as in Figure 3.