Degeneracy Breaking, Modal Symmetry and MEMS Biosensors

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Abstract: This work is concerned with degenerate modes of vibration in problems involving rotational symmetries of finite order. In the case of infinite order symmetry of the material and the geometry, two degenerate mode pairs exist for each natural number, corresponding to the cyclic periodicity of the solution. Degenerate sensors have been developed in MEMS over the past 20 years that exploit weak couplings introduced between the modes by an environmental effect of interest. In the presence of breaking of the cyclic symmetry of the problem – either in the geometry or in the substrate - the degeneracy between the modes may be broken, leading to performance degradation. Thus motivated, we investigate the effects of material and geometric symmetry breaking using COMSOL Multiphysics® and analytical techniques. The possibility of significant degeneracy breaking is demonstrated, and techniques to mitigate the problem are discussed from a systems design perspective.

Keywords: MEMS, Gyroscopes, Symmetry, Degenerate, Dynamics.

1. Introduction to degenerate modes in cyclically symmetric structures

Degenerate-mode sensors have been employed since the 1960s[1] [2]. The basic principle relies on the fact that, for problems possessing cyclic symmetry of infinite order, at least two degenerate modes always exist, with identical natural frequencies. A prototypical example is given by MEMS ring gyroscopes.

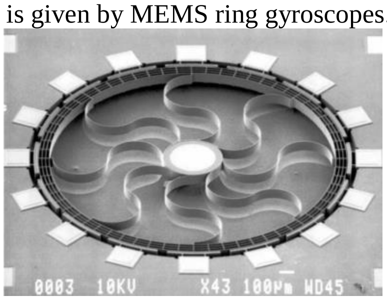

Figure 1. A cyclically symmetric ring gyroscope MEMS structure. Reproduced from[3].

These devices employ Coriolis coupling between the degenerate modes, such that one mode is driven at a controlled amplitude, and the angular velocity is inferred from the response of its degenerate partner. The importance of the degeneracy lies in the fact that the forcing experienced by the secondary mode is at the resonant frequency of the primary. To obtain the fullest response and hence to maximize sensitivity, it is necessary that the natural frequency of each mode falls within the -3dB bandwidth of its complement.

2. A Brief Characterisation of Cyclic Symmetry in Mechanical Systms

A mathematical model, of arbitrary type, capable of representing degenerate mechanical systems, is constituted by three ingredients:

- Conservation equations
- Constitutive relationships
- Boundary conditions

The first corresponds to Newton's second law for the purely mechanical case and is coupled to Maxwell's equations in the piezoelectric case, which are both well known to be symmetric under Galilean transformations.

The second corresponds to the general form of Hooke's law. For the most general case of interest, when piezoelectric effects are included, this can be written in the stress-charge form

$$= c\colon \in - e^{T}\boldsymbol{E}$$
$$\boldsymbol{D} = e\colon \in + \varepsilon \boldsymbol{E}$$

This relationship, in general, does *not* posess the full Galilean symmetry. For instance, orthotropic materials are symmetric under two or three mutually orthogonal reflections, but not for an arbitrary rotation. Thus, the symmetry of the problem may be broken by anisotropic constitutive relations.

The third ingredient in the problem is constituted by the boundary conditions. Again, in structural mechanics, the spatial extent of the geometry and the corresponding conditions can posess the full symmetry group of the space; or the symmetry may be broken. For example, a circular ring of rectangular in three dimensions retains infinite order rotational symmetry about the out-of-plane axis; attaching supporting ligaments in a symmetric configuration reduces the order of the rotational symmetry.

One analytical approach that has yielded some insight has been introduced by Gallacher. In his PhD thesis[4], the breaking of modal degeneracy in thin monocrystalline silicon rings due to the material microstructure was investigated. The anisotropic structure of the material leads to an effective periodic variation of the material properties of the ring with the angular coordinate.

In general, cyclically symmetric modes X of a cyclically symmetric system can be represented as a product of separable functions of the radial coordinate and the remainder.

Consider as a first example the simple case of a uniform thin circular ring of unity radius undergoing out-of-plane flexural vibration. The deformation in the n[th] degenerate pair of modes of vibration is described by a scalar function of the periodic scalar axial coordinate as

$$u_n^1(\theta) = A\cos(n\theta)$$
$$u_n^2(\theta) = B\sin(n\theta)$$

Where A and B are generalized coordinates for the degenerate mode pair. Then, the kinetic and potential energies can be written in the form

$$V = \frac{Ebh^3}{2}\int_0^{2\pi} u''^2 d\theta$$

$$T = \frac{\rho bh}{2}\int_0^{2\pi} u^2 d\theta$$

One way of introducing a symmetry breaking is to consider a variation in one of the material parameters. Owing to the periodic nature of the radial coordinate, the imperfection can be represented as a Fourier series. For simplicity we consider only a single component; the principle derived generalizes directly. Assume the Young's modulus is the modulated quantity. Then, we have

$$\tilde{E}(\theta) = E_0 + E_1\cos(m\theta)$$

where the phase of the modulation has been defined arbitrarily. Thus the potential energy becomes

$$V = bh^3\int_0^{2\pi} (E_0 + E_1\cos(m\theta))u^2 d\theta$$

Applying Rayleigh's quotient to the problem with the unperturbed mode pair as trial functions, we obtain

$$\omega_{nat}^2 \approx R = \frac{h^2\int_0^{2\pi}(E_0 + E_1\cos(m\theta))\cos^2(n\theta)\,d\theta}{\rho\int_0^{2\pi}\cos^2(n\theta)\,d\theta}$$

Decomposing the integral in the numerator by terms, we have

$$\omega_{nat}^2 = \omega_0^2 + \frac{E_1 h^2\int_0^{2\pi}(\cos(m\theta))\cos^2(n\theta)\,d\theta}{\rho\int_0^{2\pi}\cos^2(n\theta)\,d\theta}$$

The second term is identically zero unless the condition $n = 2m$ is met by the Fourier components of the symmetry breaking and the mode order respectively.

The treatment given above neglects the change in mode shapes incurred by the symmetry breaking. The errors incurred, and hence the frequency splitting, are quadratic in the subspace gap[5] between the new and assumed mode shapes. If the strength of the symmetry breaking is characterized by E_1, then the splitting of modes for which $n \neq 2m$ is expected to vary as $O(E_1^2)$, while the modes for which the condition $n = 2m$ is met will split as $O(E_1)$, with a much larger constant concealed by the Landau notation.

When several coupled deflection fields exist, as for more general geometries, the approach becomes more complicated. The Fourier series become matrix valued. However, the general flavor of the results still applies.

5. A Flavour of the COMSOL Multiphysics® Results

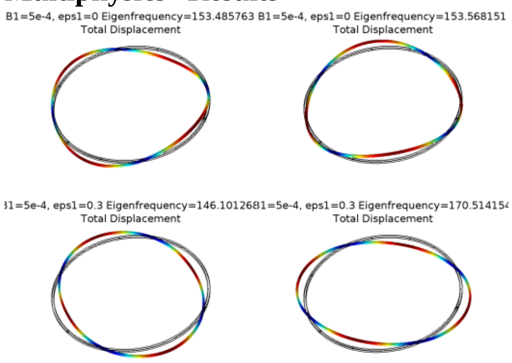

Figure 2. Detail of the degenerate out-of-plane mode pair with $n = 2$ before and after symmetry breaking [3].

A Structural Mechanics model was formulated for the thin ring discussed above. The Young's modulus was defined by the geometry-dependent expression

$$E = 70e9 \times (1 + eps1 \times (\cos(4 \times atan2(y, x))))$$

so that $m = 4$ in the above equation and the relative modulation of the stiffness is $\pm eps1$. A parametric sweep over 100 values of $eps1$ was undertaken, and the frequencies extracted. The results are presented in Figure 3:

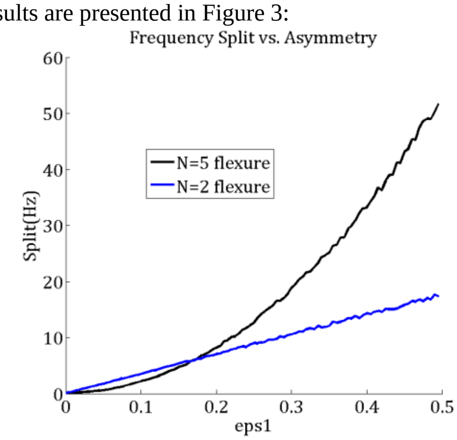

Figure 3. Frequency split plotted for two out-of-plane mode pairs against asymmetry strength.

The predicted behavior is well replicated by COMSOL Multiphysics, validating the modelling assumptions.

A particular case of practical interest is when the symmetry breaking is due to intrinsic bulk material properties, as can arise in monocrystalline materials. Our research group is currently concerned with the theory of cyclically symmetric SAW devices, to be fabricated using MEMS technologies from monocrystalline wafers.

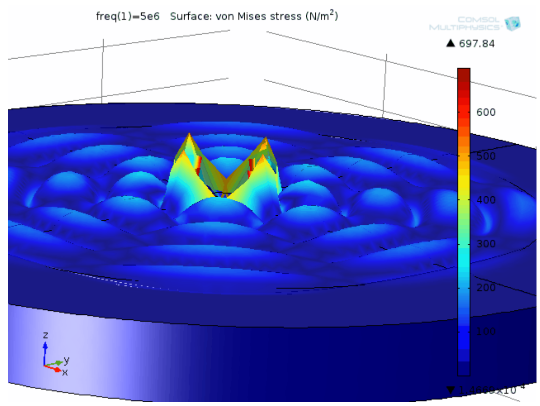

Figure 5. Time-dependent simulation of surface acoustic wave propagation on a cyclically symmetric substrate. The mode order is $n = 3$.

$LiNbO_3$ is a member of the Trigonal crystal system. Therefore, devices with cyclically symmetric geometry will have overall rotational symmetry of order $m = 3$. No mode can satisfy the condition that $n = \frac{m}{2}$, meaning that all modes will be frequency split as $O(E_1^2)$, where E_1^2 is the strength of the perturbation to the stiffness coefficients as a function of the angular parameter.

5. Conclusions

The symmetry possessed by all the fundamental physical equations can be broken, either by constitutive relations (material properties) or boundary conditions (geometric configurations). For the case of cyclically symmetric structures, symmetry breaking by the constitutive route has been considered. It was assumed that the asymmetry can be represented as a known Fourier series in the radial coordinate. In this particular case, by combining the approach of Gallacher with results from functional analysis, for a symmetry breaking of strength ε and order m, it has been demonstrated that the modes of order $m/2$ are split in the frequency domain as $O(\varepsilon)$ with a large constant, since interaction occurs between the mode shape and the form of the asymmetry, causing preferential alignment and antialignment of the degenerate pair; the other modes split as $O(\varepsilon^2)$, as effects of order ε manifest only for the perturbation of the mode shape, also of order ε.

8. References

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