An Extension of Lauwerier's Solution for Heat Flow in Saturated Porous Media

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Abstract: One of the crucial topics in this century is sustainable energy. Since the sources of fossil fuels are limited and are going to be exhausted, there is a need to look for sustainable renewable energy. In this respect, the exploitation of geothermal energy from deep hot aquifers becomes opportune. Hence, insight is required in the heat balance of potential aquifer systems. Essential issues are convection, conduction and dispersion.
Modeling such processes is affected by numerical errors when using computer models and by the complexity of analytical solutions. This article focuses on Lauwerier's problem. As an extension, it is suggested that beside convection in the aquifer and conduction to adjacent layers also conduction in an aquifer can be considered in a simple way. For a characteristic situation, a comparison is made with the result of the numerical code COMSOL. This gives new insight in the possible misjudges of heat transport simulations due to numerical effects and in the applicability of models.

Keywords: heat transfer, porous media, analytical solution, heat loss (bleeding).

1. Introduction

Geothermal energy is a promising area for substitution of fossil fuels. Recently, much effort is given to optimize production by various modeling techniques. Analytical models are useful for obtaining a general view of principle effects and numerical calculations are proper techniques for the evaluation of complex situations. However, analytical models have serious limitations considering realistic applications and numerical models hamper from numerical errors and instability. For proper understanding of these errors, and for validating the numerical models, it is useful to regularly compare analytical and numerical results.

An analytical solution for convective heat transport in porous media is given by Lauwerier (1955). An extension to Lauwerier and a complete solution for convective-conductive heat flow together with bleeding to adjacent layers has recently been developed by Barends (2009). These analytical solutions have been used to identify the numerical errors.

2. Mathematical formulation
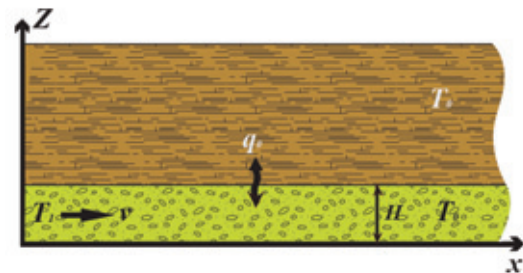
Figure 1. Schematization of the heat process in a two-layer system

The quasi two-dimensional convection-conduction heat balance equation in a plane-symmetric one-dimensional aquifer bounded by a conducting impervious adjacent layer (aquiclude) is described by following set of equations, assuming instant thermal equilibrium between fluid and grains.

$$D\frac{\partial^2 T}{\partial x^2} - v\frac{\partial T}{\partial x} + \frac{Q}{H} = \frac{\partial T}{\partial t} + \frac{h}{H}q_0$$

where $x > 0, t > 0$ (1a)

$$D'\frac{\partial^2 T}{\partial z^2} = h\frac{\partial T'}{\partial t}$$

where $z > 0, t > 0$ (1b)

Bleeding: $hq_0 = D'\dfrac{\partial T'}{\partial z}\bigg|_{z\downarrow 0}$

at the interface $z = 0, t > 0$ (1c)

Here:

$$D = \frac{\lambda}{\rho c} \quad , D' = \frac{\lambda'}{\rho c} \quad , h = \frac{(\rho c)'}{\rho c}$$

$$\rho c = \left(\phi \rho_f c_f + (1 - \phi) \rho_s c_s \right)$$

$$(\rho c)' = \left(\phi' \rho_f c_f + (1 - \phi') \rho_s' c_s' \right)$$

$$v = Rw \quad , \quad w = \frac{q}{\phi} \quad , \quad R = \frac{\phi \rho_f c_f}{\rho c}$$

Where:

- T temperature [^{o}C]
- Q heat source (heat production per meter width)
- H aquifer height [m]
- R thermal retardation factor
- T_1 injected temperature [^{o}C]
- T_0 initial temperature [^{o}C]
- λ heat conductivity[J/(ms^{o}C)]
- ρ density [kg/m^3]
- c specific heat capacity [J/(kg^{o}C)]
- w real velocity   [m/s]
- q Darcy velocity [m/s]
- ϕ porosity
- hq_0 heat flux across the interface between the aquifer and the adjacent layer [^{o}Cm/s]
- D thermal diffusivity of the aquifer [m^2/s]
- v heat velocity [m/s]

The subscript f and s refer to the porous fluid and the porous solids, and the accent refers to the adjacent layer. Flow and conduction in the aquifer is along the x-direction. Bleeding (conduction) in the adjacent layer is in the z-direction.

Flow in a porous medium induces dispersion due to scatter at smaller scale. Following the approach of Bear (2003), the thermal hydrodynamic macrodispersion is

$$D = \frac{\lambda}{\rho c} + A_L \frac{\rho_f c_f}{\rho c} \quad q = \frac{\lambda}{\rho c} + A_L v \qquad (2b)$$

consisting of the conductive part and the mechanical part. Here, A_L is the longitudinal mechanical macrodispersivity.

The conditions for the system are:

D, D', h, v, A_L, H constant   (3a)

$T = T' = 0, x > 0, \quad z > 0, t < 0$ (3b)

$T = T' = 0,$ for $(x, z \to \infty, t > 0$ (3c)

$T = T_1 U[t],$ at $x = 0$ (jump at $t = 0$)   (3d)

$Q = 0,$ for $x > 0, t > 0$ (3e)

Here U[t] is the unit step function: U = 0 for t < 0 and U = 1 for t > 0.   (2)

3. Analytical solutions

Laplace transform is applied to equation (1), without the source term, and (3).

$$D \frac{\partial^2 \Theta}{\partial x^2} - v \frac{\partial \Theta}{\partial x} = s\Theta - \frac{D'}{H} \frac{\partial \Theta'}{\partial z} \Big|_{z \downarrow 0}$$

in the domain $x > 0, s > 0$ (4a)

$$D' \frac{\partial^2 \Theta'}{\partial z^2} = sh\Theta'$$

in the domain $z > 0, s > 0$ (4b)

$\Theta = \Theta' = 0$
for $x \to \infty$ and for $(z \to \infty, x > 0)$ (5c)

$$\Theta = \frac{T_1}{s}$$

at $x = 0$ (5d)

With regard to condition (5c) the solution for equation (4b) is

$$\Theta' = \Theta \exp[-z \sqrt{\frac{sh}{D'}}] \quad \Rightarrow$$

$$D' \frac{\partial \Theta'}{\partial z} \Big|_{z \downarrow 0} = -D' \sqrt{\frac{sh}{D'}} = -\sqrt{shD'}$$

and equation (4a) becomes

$$D \frac{\partial^2 \theta}{\partial x^2} - v \frac{\partial \theta}{\partial x} = (s + \sqrt{\frac{shD'}{H^2}})\Theta \qquad (6)$$

Trial by $\Theta = B \exp[\alpha x]$ provides the so-called characteristic equation

$$D\alpha^2 - v\alpha - (s + \sqrt{\frac{shD'}{H^2}}) = 0$$

It has one positive and one negative real root.

With regard to condition (5c) the negative root applies, and the solution of (6) becomes with condition (5d) and $B = \theta = T_1 / s$

$$\Theta = \frac{T_1}{s} \exp\left[x \frac{v}{2D} \right] \exp\left[-x \sqrt{\left(\frac{v}{2D}\right)^2 + \frac{s}{D} + \sqrt{\frac{shD'}{H^2 D^2}}} \right] \tag{7}$$

The exact solution can be found by Laplace inverse transform (see Barends, 2009). In literature, Lauwerier's solutions is mentioned for a special case of (7), which is discussed and extended here.

4. Convection, conduction and bleeding

Lauwerier assumes that heat transfers in the aquifer just by convection (no conduction) and into the adjacent layers by vertical conduction. So the heat distribution in the reservoir is assumed independent vertically uniform.

In this case, equation (7) can be used with $D = 0$. The elaboration for D tending to zero with the use of Taylor expansion can be simplified and by the inverse Laplace transform (Bateman, 1954a), the solution becomes:

$$T = T_1 erfc\left[\frac{x}{H} \sqrt{\frac{hD'}{4v\left(vt - x(1-\delta)\right)}} \right] U\left[vt - x(1-\delta) \right] \tag{8}$$

where $\delta = \dfrac{hDD'}{v^2 H^2}$

The conditions related to the elaboration with Taylor expansion are:

$$t > \frac{D}{v^2} \quad , \quad t > \frac{4H^2}{hD'} \quad and \quad t > \frac{4D}{v^2} \frac{hDD'}{v^2 H^2}$$

Since the interest is focused on the behavior of the front at larger times, these conditions can be easily satisfied. The mathematical formulation is reformulated using following dimensionless variables:

$$\chi = \frac{xhD'}{vH^2} \quad , \quad \tau = \frac{thD'}{H^2}$$

$$T = T_1 \, erfc\left[\frac{\chi}{2\sqrt{\tau - \chi(1-\delta)}} \right] U[\tau - \chi(1-\delta)] \tag{9}$$

The Lauwerie solution is similar to equation (9) when adopting $\delta = 0$. Formula (8) is reformulated for a specific reference temperature T_0 and injected temperature of T_1 according to:

$$\frac{T - T_0}{T_1 - T_0} = erfc\left[\frac{x}{H} \sqrt{\frac{hD'}{4v\left(vt - x(1-\delta)\right)}} \right] U\left[vt - x(1-\delta) \right] \tag{10}$$
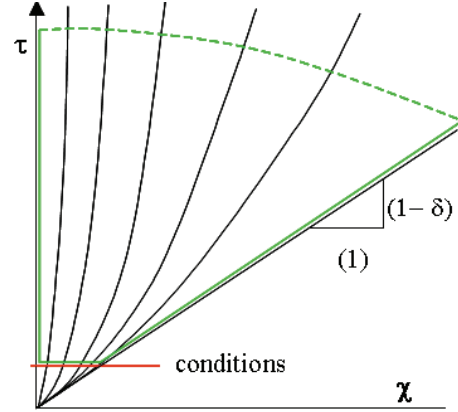

Figure 2. Isothermal lines in the (χ,τ) plane for equation (9).

Figure 2 shows the effect of δ in the extended Lauwerier solution, equation (9).

Next, the parameter effect consider using the equation (10).To be able to see the effects more clear each parameter has been changed 1, 2, 5 and 10 times. Some of these values may be are not realistic.
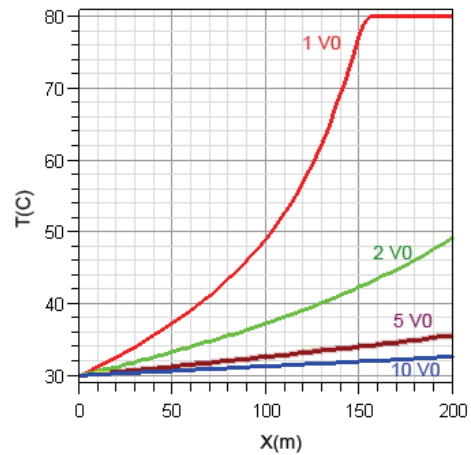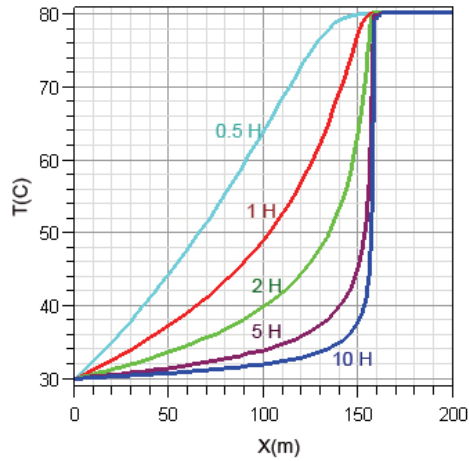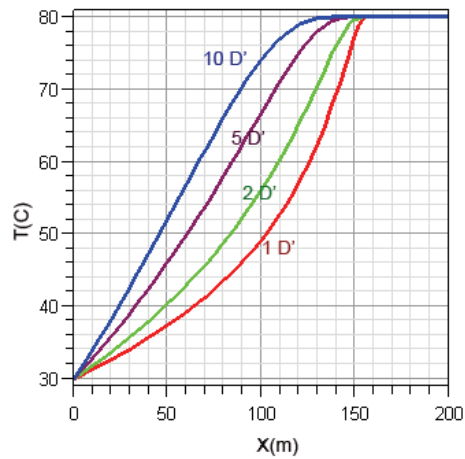

Figure 3. Graphs of equation 10 for v/v_0=1, 2, 5, 10; v_0=1E-7[m/s]

Figure 3 shows the sensitivity of the solution to fluid velocity. As expected, convection has a large effect on the breakthrough time.
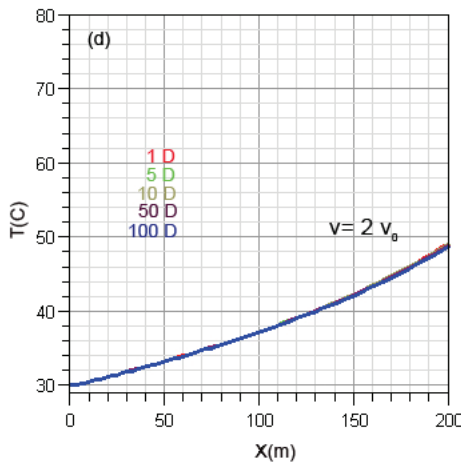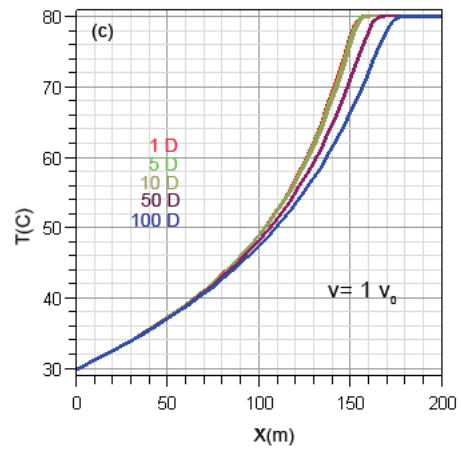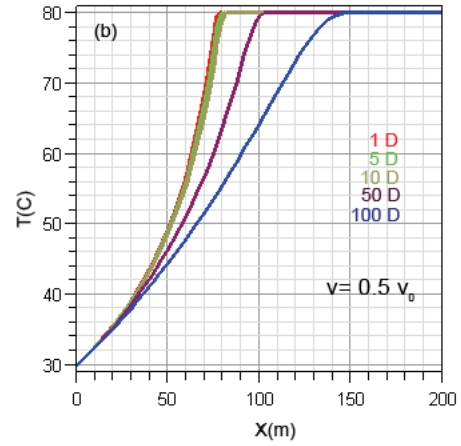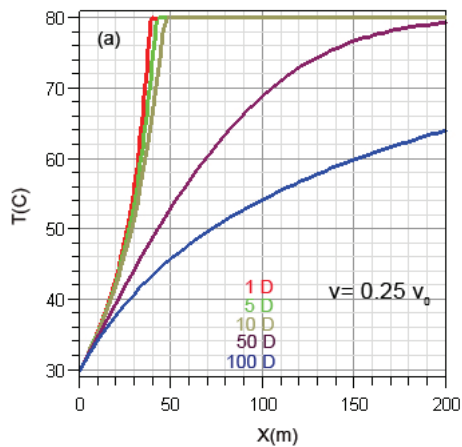
Figure 4. Effect Graphs of equation 10 for H/H_0=0.5, 1, 2, 5, 10; H_0=15 m

Heat loss (bleeding) is more affective in thin aquifers, figure 4.

Figure 5. Graphs of equation 10 for D'/D'_0=1, 2, 5, 10; D'_0= 8.9E-7

Figure 6. Effect of conductivity of the aquifer on heat transport D/D_0=1, 5, 10,50,100. Each graph show D effect in different velocities

As the thermal conductivity of the top layer increase the bleeding effect becomes higher (figure 5). In the case of cold water injection to the hot aquifer, top layer cools off

while heating the aquifer which became colder duo to injected water.

Graphs in figure 6 show the reservoir conductivity effect on heat transport. When convection (velocity) increases thermal diffusivity effects become less dominant, even vanish.

5. Comparison with COMSOL Multiphysics

The numerical calculations have been elaborated using Comsol Multiphysics 3.5. Here, in a hot aquifer with original temperature of $T_0 = 80$ degrees Celsius, confined between two impermeable layers, a hot water doublet system is installed, the filters placed at a distance of 200m. Cold water of $T_1 = 30^oC$ will be injected. The numerical simulation shows how and when cold water reaches the pumping well (breakthrough). The following assumptions were considered:

1. The aquifer is homogeneous and infinite in horizontal direction; its thickness H is constant.
2. The caprock and the bedrock, above and below the aquifer, are homogeneous and impermeable.
3. The aquifer is located at the 2000 m depth.
4. Fluid flow in the aquifer is assumed to be steady; injection rate Q is equal to the production rate.
5. Wells fully penetrate the aquifer.
6. Thermal equilibrium is supposed to take place instantaneously between the water and the rock matrix in the aquifer.
7. Volumetric heat capacity (ρc) for both the water and the rock, and the caprock, are constant.
8. Differences in viscosity between injected water and initial water are disregarded.
9. Heat loss (bleeding) occurs only through top layer (caprock). The lower boundary is considered a thermal isolator.
10. The thermal and hydraulic properties of the ground are estimated from a realistic situation (appendix).
11. Aquifer thickness is kept small in order to better compare with the 1D analytical solution.
12. The distance between injection and pumping well is 200 m which is smaller than reality, for reasons of calculation time.
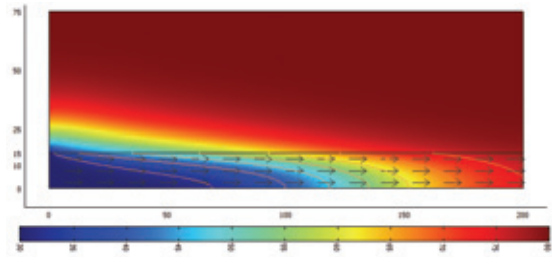

Figure 7. Temperature field after 5 year injection of cold water. Arrows show flow direction in the aquifer
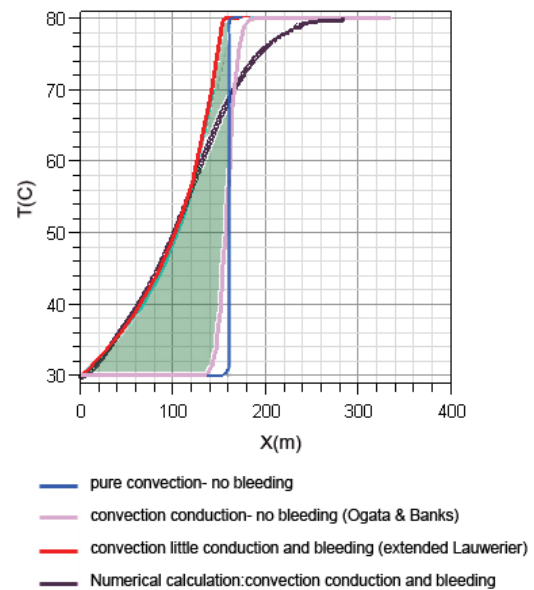

- pure convection- no bleeding
- convection conduction- no bleeding (Ogata & Banks)
- convection little conduction and bleeding (extended Lauwerier)
- Numerical calculation:convection conduction and bleeding

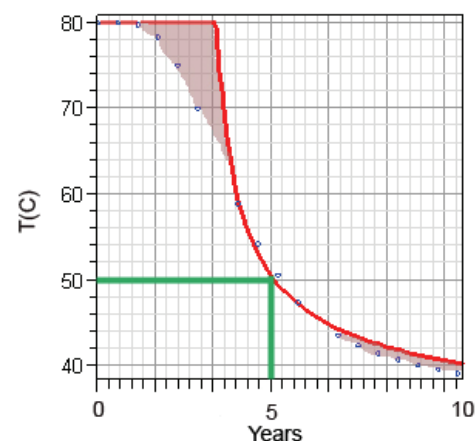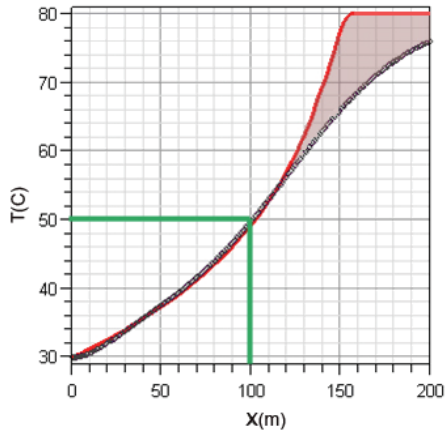**Figure 8. Comparison of different analytical methods with numerical result**

Figure 9. Breakthrough curve at x = 100 m. Red: analytical solution (12), blue: numerical calculation.

Figure 10. Temperature distribution after 5 year. Red: analytical solution (12), blue: numerical calculation.

Figure 8 compares several analytical solutions for different conditions with numerical results. The blue line shows pure convection in the aquifer with a sharp front. The pink line demonstrates the solution of Ogata and Banks (1961) which deals with convection and conduction in the aquifer without bleeding. The pink line can perfectly illustrate the conduction effect when compared with the blue line. The red line is the extended Lauwerier solution. The green painted area between these lines shows the bleeding effect which is significant. Numerical result by COMSOL is the brown line.

Figure 9 and 10 show the numerical result compared to the extended Lauwerier solution. The difference is painted in pink. It is due to numerical error (numerical dispersion) and a mismatch of the semi-2D analytical solution and the full 2D numerical solution.

Numerical errors in models can lead to mis-evaluation of the geothermal reservoir life time and heat potential which is of economic importance.

6. Conclusion

The extended Lauwerier solution which considers both conduction-convection in the aquifer and conduction in the adjacent layer (bleeding) shows that conduction in the aquifer is significant especially in the case of low flow velocity. A comparison which was made between a numerical calculation by COMSOL and the extended Lauwerier solution shows a serious deviation which is mainly caused by numerical dispersion.

There are methods to decrease numerical errors but in general errors are inevitable. An analytical solution can provide an idea about the size of numerical errors. Understanding of numerical errors can provide more accurate interpretation of numerical simulations.

7. References

1. Barends, F.B.J. ,Heat convection, conduction and bleeding in porous media; Analytical solutions, Private Communication (2009).
2. Bateman H. ,Tables of integral transforms.,**Vol I**, pg 245, equ. 3. McGraw-Hill BC, London (1954)
3. Bear, J. (2003) Mathematica models of flow and contaminant transport in saturated porous media. Proc. Modelling Coupled Phenomena in Saturated Porous Materials (ed Kubic, Kaczmarek & Murdoch). Polish Acad of Scienecs, Warzaw, Poland
4. Lauwerie H.A., The transport of heat into an oil layer caused by the injection of hot fluid. *J.Appl.Sc.Res*., Section A, **Vol 5**, Nr 2-3, pag 145-150 (1955).
5. Ogata A. and Banks R.B., A solution of the differential equation of longitudinal dispersion in porous media, *U.S. Geological Survey,* Professional Paper **vol. 411-A** (1961).

8. Appendix

Table 1: The parameters used for numerical calculations

ϕ	0.25	porosity
k	2.8E-13[m^2]	permeability
ρ_f	1200[kg/m^3]	Saline warm fluid density
ρ_s	2800[kg/m^3]	solid density
λ_s	1.7[W/(m.degC)]	Heat conductivity of rock
λ_f	0.6[W/(m.degC)]	Heat conductivity of fluid
H	15[m]	thickness
T_c	30[degC]	cold temperature
T_f	80[degC]	warm temperature
c_f	4184[J/kg/degC]	specific heat capacity of fluid
c_s	830[J/kg/degC]	specific heat capacity of solid
k_t	1E-18[m^2]	permeability of top layer
ϕ_t	0.05	porosity of top layer
ρ_t	3000[kg/m^3]	density of top layer
λ_t	2.6[W/(m.degC)]	thermal conductivity of top layer
c_t	840[J/kg/degC]	heat capacity of top layer
T_t	80[degC]	top layer temperature