Determination of Constitutive Properties using DIC-Displacement Data and U-FEM

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Introduction

A new inverse problem formulation for identification of constitutive parameters in a side-notched paperboard composite loaded vertically from measured displacement data is developed using Levenberg-Marquardt Algorithm (LMA) and finite element method. Displacements were recorded using digital image correlation (DIC). The cost function, which is the difference between the measured DIC v-displacement and the reconstructed displacement from finite element method, was minimized using LMA. The primary advantage of this new formulation is the direct use of displacement data and eliminating the need for numerical differentiation if strain or stress data were used. The inverse method algorithm determined the constitutive properties with errors smaller than 10%. Selection of initial conditions and comparison with other inverse methods will be addressed.

One method of evaluating constitutive properties of orthotropic materials is the use of inverse methods (IM). Avril and Pierron [1] reviewed several IM approaches and showed their general equivalency. IM can be generally described as the iterative adjustments of parameters (constitutive properties) in a numerical model (updated finite-element scheme) to minimize the difference between an experimentally measured quantity (displacement) and the numerically calculated quantity. By comparing calculated out-of-plane displacement from Finite Element Method (FEM) with those measured by shadow moiré, Le Magorou et al. [2] determined bending/torsion rigidities in composite wood panels by the resolution of IM. Molimard et al. [3] evaluated constitutive properties of a composite material by minimizing the difference between moiré-measured displacements and those predicted by FEM in a perforated tensile plate. Similarly, Genovese et al. [4] used IM procedures to evaluate a truss system and a composite plate where Considine et al. [5] determined material properties in heterogeneous materials from full-field simulated displacement data. Each of these references incorporated a specific type of IM entitled FEMU-U (finite element method updating displacement). The root mean square of displacement differences between the measured values and those predicted by FEM were minimized by iteratively changing the constitutive properties in the FEM model. FEMU-U is attractive because displacements are first-order outputs of high-resolution full-field techniques of DIC where strain is a second-order output and has greater noise associated with numerical differentiation. Alshaya et al. [6] determined the constitutive properties of a symmetrically sided notched graphite/epoxy composite using recorded DIC displacement data and Airy stress function. Reference [7] reviews recently developed methods for constitutive parameter identification based on kinematic full-field measurements, namely the finite element model updating method (FEMU), the constitutive equation gap method (CEGM), the virtual fields method (VFM), the equilibrium gap method (EGM) and the reciprocity gap method (RGM).

The mechanical properties of composite plate specimens can be also evaluated by using of experimental eigenfrequencies, the corresponding numerical eigenvalue evaluation, sensitivity analysis and optimization [8]. The constrained minimization of an error functional expressing the difference between measured higher frequencies of a plate specimen and the corresponding numerical ones is then carried out to find the desired optimum parameters.

In 2-D models, the degree of freedom is (number of nodes) × 2 - (number of constitutive parameters) - 1. For homogeneous, isotropic materials, the number of constitutive properties is two $(E;\nu)$; for homogeneous, orthotropic materials, the number of constitutive parameters are four $(E_1;E_2;G_{12};\nu_{12})$. For either case, the number of degrees of freedom is large and the problem is solved by minimizing least squares of the chosen cost function.

The goal of this work is to evaluate the constitutive properties of orthotropic composite materials using IM and COMSOL LiveLink. Prior to using actual measured displacement data, a simulated v-displacement from COMSOL was used to verify the reliability and robustness of the method. The primary advantage of the proposed inverse method is the fact that only one in-plane displacement was used even though DIC provides the two in-plane displacements.

Therefore, the proposed IM determines the constitutive properties from only a single experimental test. The use of a COMSOL LiveLink for MATLAB facilitated the passing of information to and from the optimization algorithm, and allowing for streamlined, efficient and accurate of the LMA. The authors are unaware of prior utilizing of DIC displacement in conjunction with COMSOL LiveLink to experimentally determine the constitutive properties in composites.

Problem Definition and Governing Equations

The governing equations for anisotropic elasticity consist of the equilibrium equations for static loading conditions, the strain-displacement relations for small deformations as well as the stress-strain laws for linear anisotropic elastic solids,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad i = 1,2,3$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i,j = 1,2,3$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \qquad i,j,k,l = 1,2,3$$

Where $\sigma_{ij} = \sigma$ is the stress tensor, $\varepsilon_{ij} = \varepsilon$ is the strain tensor, $u_i = \mathrm{u}$ is the displacement vector, C_{ijkl} is the 4th-order stiffness tensor, and f_i is the body forces. These three equations sets constitute 15 partial differential equations with 15 unknown functions $u_i, \varepsilon_{ij}, \sigma_{ij}$, $i,j = 1,2,3$, in terms of three spatial coordinate variables x_i, $i = 1,2,3$. If the geometry and material properties are known, the equations sets can be solved. However, if, for instance, the displacements are known in the domain, the material properties can be determined using inverse approach.

The developed inverse approach is utilized to analyze a finite-width tensile paperboard orthotropic plate (from US Forest Products Laboratory, Madison, WI; $E_x = E_2$ = 2.12 GPa, $E_y = E_1$ = 4.52 GPa, $G_{xy} = G_{12}$ = 1.27 GPA, $\nu_{xy} = \nu_{21}$ = 0.18(4.52)/2.12 = 0.3838) with side notches of radius R = 13.5 mm. The plate was loaded in the strongest/stiffest material direction (1-, y-direction), Figure 1. Over-all laminate dimensions are 225 mm long, 63.5 mm wide and 0.31 mm thick. The coordinate origin is at the center of the plate and the response is symmetric about x- and y-axes. The laminate elastic properties were obtained from conducting uniaxial tensile tests in the strong/stiff (y-direction), weak/compliant (x-direction) and 45-degree orientations.

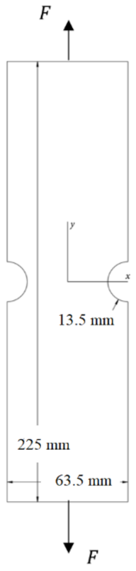

Figure 1. Vertically-loaded finite paperboard composite plate with circular side notches.

Digital Image Correlation

Digital Image Correlation (DIC) is a full-field computer-based image analysis technique for the non-contact measurement of displacements of a surface equipped with a speckle pattern. The method tracks the motion of the speckles by comparing the gray scale value at a point (subset) in a deformed and undeformed configuration. Two sets of images are recorded; the first image typically being at zero load and the second image under load. A single camera setup can record both u and v in-plane full-field deformations. Out-of-plane motions can also be recorded if two cameras are employed. The achievable DIC resolution depends on a number of factors, including but not limited to, camera resolution, lens optical quality, and speckle size and quality. Unlike electronic speckle, DIC necessitates the surface under study to have a speckle pattern for tracking; but unlike thermoelastic stress analysis, DIC does not require cyclic loading.

MatchID software was used to record the images of the plate in its loaded and unloaded conditions and to evaluate the displacements for post-processing. When utilizing two cameras, a separate calibration grid was used to evaluate the displacement data in physical units. Quality displacement information at and near the edge of the notch and at (near) the longitudinal edge of the specimen is unavailable because the DIC software's correlation algorithm is unable to track a group of pixels (subset) which lack neighboring pixels. To perform the tracking, the subset is shifted until the pattern in the deformed image closely matches that of the reference image.

An optimized random gray level pattern was produced on a sheet of the material, nominally 205 mm × 250 mm, using a Sharp® (Sharp electronics Corp., Mahwah, NJ) MX-3100N copier. Specimen was cut from the sheet after preconditioning and conditioning. The DIC pattern is shown on a specimen in Figure 2. The plate was statically loaded in the hydraulic grips of a 20 kips capacity MTS hydraulic testing machine from 0 N to 480 N in 30 N load increments. Displacement data were recorded and processed at each load increment. Before conducting the quantitative analysis, two cameras were used to capture the three displacement components by which to verify there was no out-of-plane bending.

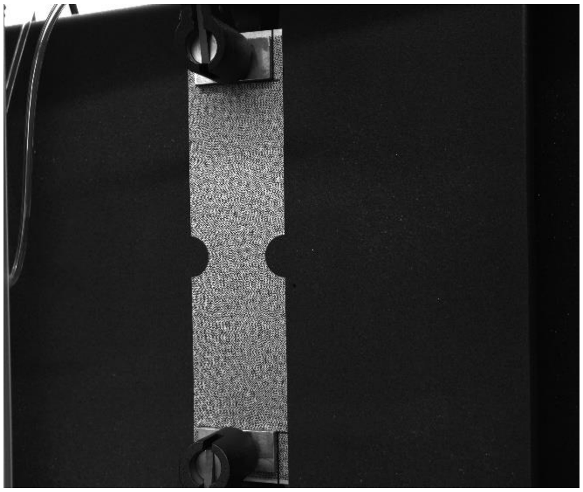

Figure 2: DIC pattern in the vertically-loaded finite paperboard composite plate with circular side notches.

The recorded displacement information was exported to MATLAB® (Mathworks, Inc., 2015) to convert each pixel into a data point, i.e., points in Figure 3. Since DIC data typically are unreliable on and near an edge, no recorded displacements were used within at least 3.63 mm of the boundary of the notch.

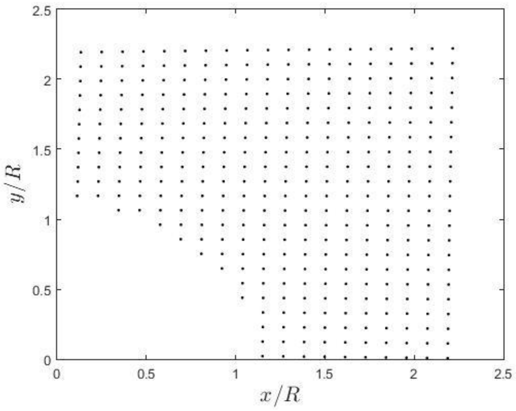

Figure 3: Source locations of the selected v-displacement from DIC.

The DIC MatchID software provided approximately 8,160 values of u and v when the analysis was carried out for 21 subsets in 10 steps. The plate is geometrically and mechanically symmetrical about the vertical y-axes, Figure 1. Since the top end of the physically tested plate was fixed stationary while the bottom end moved vertically downward, the zero vertical displacement was shifted to be at the horizontal middle of the plate to represent the case of the plate being extended at both top and bottom ends. The measured v-displacement data were subsequently averaged about all quadrants to cancel any asymmetry and the resulting averaged measured of values of 342 v-displacement data are plotted in the second quadrant as shown in Figure 4 (right). Due to the previously mentioned unreliability, recorded data on and near the edge of the notch were not employed.

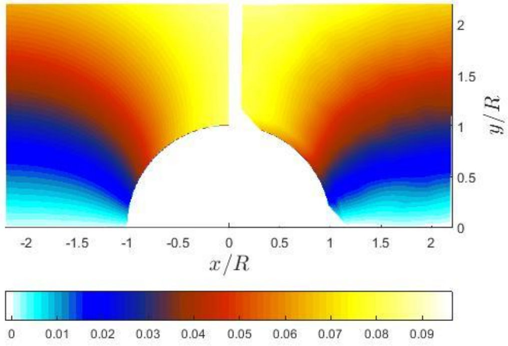

Figure 4: Averaged DIC v-displacement (right) and COMSOL v-displacement (left) when constitutive properties were used.

The measured DIC data were digitized in matrix form and combined with COMSOL LiveLink to determine the constitutive properties using LMA as will be discussed in the subsequent sections.

Inverse Method Procedure

The particular inverse method used here determines the constitutive properties of the paperboard through an iterative process by minimizing the difference between the reconstructed v-displacement from COMSOL finite element software and the DIC measured v-displacement. Therefore, the function to be minimized is

$$f(\hat{u}_{FEM}, P) = \|r\|, \quad \text{where} \quad r = \hat{v}_{DIC} - \hat{v}_{FEM}$$

where $\hat{v}_{FEM}$ and $\hat{v}_{DIC}$ are vectors containing nodal v-displacements determined by COMSOL and DIC, respectively, P is a vector containing the constitutive parameters, E_x, E_y, G_{xy}, and ν_{xy}, and $\|r\|$ is the norm of r. Because the previous equation is nonlinear with respect to P, iterative procedures are necessary to minimize $f(\hat{v}_{FEM}, P)$ and determine P. Levenberg-Marquardt Algorithm (LMA) is commonly used because it combines the benefits of Steepest Descent Method and Gauss-Newton Method. The LMA has the following form

$$P_{i+1} = P_i - \left(J^T J + \lambda \cdot \mathrm{diag}(J^T J)\right)^{-1} J^T r$$

where i is iteration number, J and J^T are Jacobian and Jacobian transpose, determined by backward difference, $J(m,n) = \partial r_m / \partial P_n$; m is the number of nodal displacements and n is the number of constitutive parameters (4 in this work), and λ is a non-negative damping factor, adjusted each iteration, to alternate between Steepest Descent and Gauss-Newton Methods.

The primary disadvantage of LMA is the need for matrix inversion during each iteration. In most applications, reduced iterations compensate for the matrix inversion. After calculating a new P_{i+1}, the constitutive parameters are checked for validity, i.e., a positive-definite stiffness matrix, and are adjusted if not valid. The validated P_{i+1} are inputs to a new analysis and the resulting nodal displacements are used to determine f_{i+1}. If $f_{i+1} < f_i$, the constitutive parameters are updated, $P_{i+1} \rightarrow P_i$, λ is reduced by a factor of 10, and the next iterations begins. If $f_{i+1} > f_i$, then λ is increased by a factor of 10 and P_i is not updated. As $\lambda \rightarrow 0$, LMA becomes exactly the Gauss-Newton Method. The iterations are processed until the errors between two consecutive results of each constitutive properties are less than a user-defined stopping criteria (ε_s = 0.01\% in this work),

$$\max_{\forall j}\left\{\left|\frac{P_{i+1}(j) - P_i(j)}{P_{i+1}(j)}\right| \times 100\right\} < \varepsilon_s, \qquad j = 1,2,3,4$$

LMA requires two initial estimates of P in order to calculate J and begin iterations. Genovese et al. [4] evaluated the effect of initial estimates on the number of iterations using FEMU-U in an overdetermined system and found that poor initial estimates increased the iterations required for minimization, but minimization was eventually achieved. Although evaluation of initial estimates is beyond the scope of this investigation, an informal analysis showed that randomly assumed initial guesses lead eventually to convergence of the constitutive properties. The first initial estimates for each of the constitutive properties were the same with the expected values. The second initial estimates were randomly assumed with at most two times larger or smaller than the expected values. The effect of changing the initial estimates is addressed in the subsequent sections.

Numerical Model and Simulation Steps

The governing equations are discretized using the finite element method by way of COMSOL Solid Mechanics. A finite element analysis (FEA) using COMSOL Multiphysics® (COMSOL, Inc, version 5.3) was prepared of the paperboard plate of Figure 1. Due to symmetry, only one quarter of the plate was modeled with symmetrical boundary condition applied at the bottom and right edges. A far-field stress of $\sigma_0 = F/A = 154/(63.5 \times 0.31) = 7.82$ MPa was applied numerically to the top edge. A very fine mesh was used in the neighborhood of the circular notch. A convergence test was applied until the change in magnitude of the maximum stress between two successive meshing was less than 2%. The FE quarter model utilizes 11352 elements.

Figure 5: Mapped-meshed FE model with dense meshing adjacent the circular side notch.

The in-plane DIC v-displacement components, $\hat{v}_{DIC}$, in Figure 4(right) were used to determine the constitutive properties of a notched paperboard loaded in strongest/stiffest direction. A MATLAB subroutine based on COMSOL LiveLink was programmed with $E_{\mathrm{x}}, E_{\mathrm{y}}, G_{\mathrm{xy}}, v_{\mathrm{xy}}$ as inputs and the constructed v-displacement as an output, $\hat{v}_{FEM}$. To initiate the model, initial guesses of the constitutive properties, $E_{\mathrm{x}}^0, E_{\mathrm{y}}^0, G_{\mathrm{xy}}^0, v_{\mathrm{xy}}^0$, i.e, P_0, were assumed. These values were used as inputs for the MATLAB subroutine to construct the corresponding v-displacement component, $\hat{v}_{FEM}$. The difference between the reconstructed v-displacement from COMSOL finite element software, $\hat{v}_{FEM}$, and the DIC measured v-displacement, $\hat{v}_{DIC}$, $r = \hat{v}_{DIC} - \hat{v}_{FEM}$, was minimized by means of LMA by choosing a new set of elastic constants. This set of elastic constants were then used again as inputs for the MATLAB routine to construct a new v-displacement, $\hat{v}_{FEM}$, which can be used to predict also a new set of elastic constants. The process was repeated until the difference percentage between two consecutive set of elastic constants is less than a user-defined stopping criteria ε_s.

Results and Discussion

A set of 516 random DIC-measured v-displacement data points, Figure 3, was used to determine the four material properties of the paperboard (E_{x}, E_{y}, G_{xy}, and v_{xy}). Different initial guesses were used to initiate the inverse method and determine the constitutive properties. Even though the DIC provides the two in-plane displacement components, only one single displacement component, the vertical v-displacement, was used to determine the constitutive properties.

Needing only a single displacement field is advantageous. For example, experimental techniques such as moiré, holography, grids or electronic speckle pattern interferometry (ESPI) necessitate additional rulings and/or optics to record two versus one in-plane component of displacement. Situations can also occur where there is a paucity or inferior quality of one or other of the measured in-plane displacements. Furthermore, such displacement-based techniques necessitate differentiating the recorded data, something which can be unreliable, especially on the edge of the geometric discontinuity and may lead to highly inaccurate strains and stresses. This latter challenge is overcome here in that the displacement data at, and in the neighborhood of the notch were not used for obtaining the constitutive properties of the paperboard and only the measured information originated away from the edge of the notch.

As shown in Figure 6, the proposed method converges to the average values of the paperboard properties (determined experimentally) regardless of the initial guesses. Whether the initial guesses are overestimated or underestimated the average experimental values of the properties, the inverse method will converge to these experimental determined values of the paperboard constitutive properties.

Monte Carlo method was used to test the reliability of the proposed inverse method and the effect of the initial guesses. To initiate the model, a domain $\mathbb{D}$ of possible constitutive properties inputs was created from the following equation

$$\begin{Bmatrix} E_x^0 \\ E_y^0 \\ G_{xy}^0 \\ v_{xy}^0 \end{Bmatrix} = E \begin{Bmatrix} R_1 E_x \\ R_2 E_y \\ R_3 G_{xy} \\ R_4 v_{xy} \end{Bmatrix}$$

where $\{E_x^0, E_y^0, G_{xy}^0, v_{xy}^0\}^T \in \mathbb{D}$ are the initial input guesses of the inverse model, E is the maximum absolute random error (user specified), R_1, R_2, R_3, and R_4 are independent generated random numbers ($0 \leq R_i \leq 1,\ i = 1,2,3,4$) and $\{E_x, E_y, G_{xy}, v_{xy}\}^T$ are the averaged experimental determined constitutive properties, E_x= 2.12 GPa, $E_y =$ 4.52 GPa, $G_{xy} =$ 1.27 GPA, $v_{xy} =$ 0.3838 in this work. For this numerical experiment, an absolute error of $E =$ 50, 100, 200 and 400% were used and 10, 20, 50, and 100 different inputs were generated randomly from the uniform probability distribution over the domain $\mathbb{D}$ to perform the deterministic computation on these inputs. The resultant outputs were aggregated, and the corresponding averages (means) and standard deviation are listed in Table 1. It shows that, regardless the initial guesses and how it far from the target values, the method will converge to the target values.

The authors are unaware of any previously published work in determination of the constitutive properties of a loaded orthotropic composite from recorded information of a single displacement field and updated-FEM using COMSOL LiveLink.

Conclusions

In this work, the constitutive properties of the paperboard composite were determined in inverse method using one in-plane DIC displacement and COMSOL LiveLink. It should be noted that the proposed method is not only restricted to loaded-notched plate but can be applicable for any geometrical and loading conditions. The use of a COMSOL LiveLink for MATLAB facilitated the passing of information to and from the optimization algorithm, and allowing for streamlined, efficient and accurate of the LMA.

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References

[1] S. Avril and F. Pierron, “General framework for the identification of constitutive parameters from full-field measurements in linear elasticity,” *Int. J. Solids Struct.*, vol. 44, no. 14–15, pp. 4978–5002, Jul. 2007.

[2] L. Le Magorou, F. Bos, and F. Rouger, “Identification of constitutive laws for wood-based panels by means of an inverse method,” *Compos. Sci. Technol.*, vol. 62, no. 4, pp. 591–596, Mar. 2002.

[3] J. Molimard, R. Le Riche, A. Vautrin, and J.-R. Lee, “Identification of the four orthotropic plate stiffnesses using a single open-hole tensile test,” *Exp. Mech.*, vol. 45, no. 5, pp. 404–411, 2005.

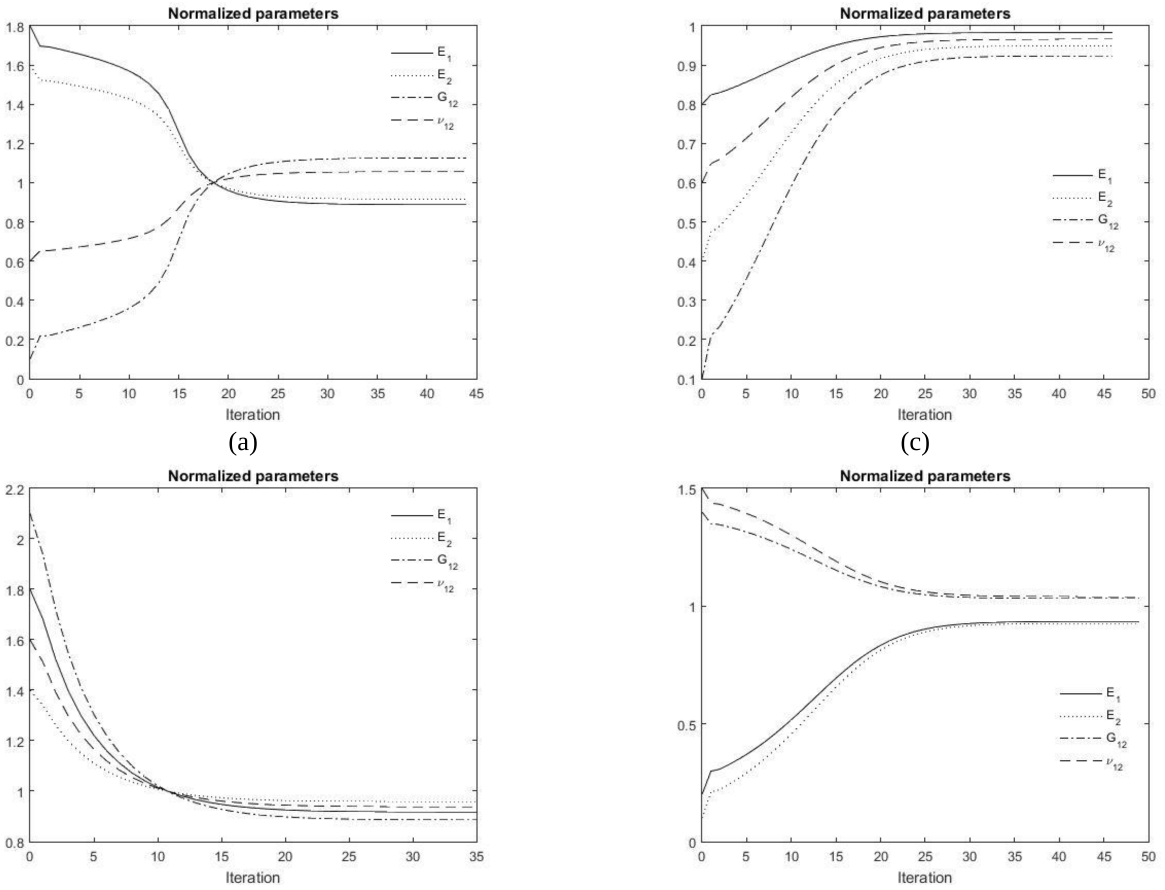
[4] K. Genovese, L. Lamberti, and C. Pappalettere, “Improved global–local simulated annealing formulation for solving non-smooth engineering optimization problems,” *Int. J. Solids Struct.*, vol. 42, no. 1, pp. 203–237, 2005.

[5] J. M. Considine, D. W. Vahey, D. Matthys, R. E. Rowlands, and K. T. Turner, “An Inverse Method for Analyzing Defects in Heterogeneous Materials,” in *Application of Imaging Techniques to Mechanics of Materials and Structures, Volume 4: Proceedings of the 2010 Annual Conference on Experimental and Applied Mechanics*, T. Proulx, Ed. New York, NY: Springer New York, 2013, pp. 339–346.

[6] A. Alshaya, J. M. Considine, and R. Rowlands, “Determination of Constitutive Properties in Inverse Problem Using Airy Stress Function,” in *Residual Stress, Thermomechanics & Infrared Imaging, Hybrid Techniques and Inverse Problems, Volume 8: Proceedings of the 2017 Annual Conference on Experimental and Applied Mechanics*, A. Baldi, J. M. Considine, S. Quinn, and X. Balandraud, Eds. Cham: Springer International Publishing, 2018, pp. 73–81.

[7] S. Avril *et al.*, “Overview of Identification Methods of Mechanical Parameters Based on Full-field Measurements,” *Exp. Mech.*, vol. 48, no. 4, p. 381, Jul. 2008.

[8] C. M. M. Soares, M. M. de Freitas, A. L. Araújo, and P. Pedersen, “Identification of material properties of composite plate specimens,” *Compos. Struct.*, vol. 25, no. 1, pp. 277–285, Jan. 1993.



(a) (c)

(b) (d)

Figure 6: The convergence of the inverse method using different initial guesses.

Table 1: Predicted values of constitutive properties of paperboard using inverse method and COMSOL LiveLink for different number of generated initial guesses.

# It.	Error (%)	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	ν_{xy}
10	50	2.21 ± 0.3110	4.23 ± 0.0769	1.33 ± 0.1787	0.4007 ± 0.0658
20		2.20 ± 0.3258	4.26 ± 0.1035	1.26 ± 0.1408	0.3931 ± 0.0293
50		2.29 ± 0.4498	4.28 ± 0.1283	1.29 ± 0.1802	0.3928 ± 0.0580
100		2.14 ± 0.3661	4.29 ± 0.1270	1.27 ± 0.2300	0.3745 ± 0.0865
10	100	2.18 ± 0.2426	4.28 ± 0.1151	1.31 ± 0.1553	0.3702 ± 0.0427
20		2.19 ± 0.5010	4.19 ± 0.3907	1.33 ± 0.1449	0.3935 ± 0.0634
50		2.25 ± 0.5233	4.24 ± 0.2705	1.31 ± 0.2294	0.3936 ± 0.0573
10	200	1.94 ± 0.3848	5.59 ± 2.9432	1.62 ± 0.5192	0.4718 ± 0.2119
20		2.14 ± 0.9159	4.90 ± 2.0482	1.54 ± 0.5450	0.4519 ± 0.1707
50		2.21 ± 0.7392	4.97 ± 2.0956	1.38 ± 0.6429	0.4459 ± 0.1879
10	400	2.07 ± 0.2670	4.28± 0.0820	1.26 ± 0.2541	0.3685 ± 0.0266
20		2.42 ± 1.3748	4.35± 6.5064	2.69 ± 2.1909	0.7786 ± 0.5229
50		2.19 ± 1.4226	4.45± 6.9325	1.29 ± 2.6975	0.3685 ± 0.5964

Target values: E_x = 2.12 GPa, E_y = 4.52 GPa, G_{xy} = 1.27 GPA, ν_{xy} = 0.3838.