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Boundary Element Technique in Petroleum Reservoir Simulation

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Abstract: Petroleum reservoir simulation is a process of modeling the complex physical phenomena inside a reservoir. The goal is to determine how hydrocarbons and water behave and how local reservoir characteristics affect the oil and gas recovery in the reservoir. This study presents an application of two rigorous analytical based numerical schemes, so called the Boundary Element Method (BEM) and its hybrid form, the Dual Reciprocity Boundary Element Method (DRBEM). They are proven to be able to provide a computationally efficient means of handling single and multi-phase flow in a homogeneous medium. The accuracy can be further enhanced by incorporating singularity programming and Laplace Transformation (LT) techniques; hence an alleviation of numerical errors caused by singularities and a time derivative is achieved. It is observed that these two methods can be an alternative tool to analyze pressure transient performance in both single and multiphase flow, which plays a key role in enhanced oil recovery processes.

Keywords: pressure, saturation, numerical, reservoir, phase

1. Introduction

In the oil industry reservoir simulation is the study of how fluids flow and behave in a reservoir under different conditions. It consists of the conceptual model which describes the relevant physical process verbally; a mathematical model which usually involves coupled or decoupled systems of nonlinear partial differential equation describing the process quantitatively, and the numerical model especially with the advent of fast computers and due to the limitations and assumptions made through an analytical approach. Reservoir simulators can estimate production characteristics, calibrate reservoir parameters; visualize reservoir flow patterns, hence they are widely used to aid the planning and implementation of enhanced oil recovery strategies. Advanced drilling techniques and enhanced seismic and geological characterization of reservoirs have emerged and has resulted in more accurate geological information [1]. Consequently, there is a substantial research activity that aims toward faster, more robust, and more accurate reservoir simulators [2, 3, and 4]. In this work we seek accurate numerical schemes combined with suitable mathematical models for the reservoir.

For single phase flow, the BEM has been an established tool for reservoir characterization through pressure transient analysis due to its rigorous analytical base of Green's function. It was indicated that the domain methods suffer from numerical dispersion and grid orientation effects due to the domain discretization [5]. Several attempts to alleviate the aforementioned problems have been reported. Examples are the flux limiter method [6], and the total variation diminishing mid-point scheme [7, 8]. Another shortcoming of the domain discretization lies in the well treatment because the flowing bottom-hole-pressure of a well is usually related to the pressure calculated for that particular grid block containing the well through well models. None of the applicable models can eliminate the steady state/pseudo steady state. One example is the popular Well Index approach [9]. On the contrary, the BEM can overcome the numerical dispersion and grid orientation effects introduced by domain discretization and remove the prominent assumptions for well treatment [10] because of the boundary-only discretization nature. The BEM was first introduced into petroleum engineering problems in 1973 [11] which demonstrated the power of Green's functions in relation to a solution for well testing problems. The success of this approach attracts many researchers who try to pursue boundary element solutions with respect to general well testing problems since this approach is able to deal with arbitrary boundary shapes and more flexible conditions through the discretization of the boundary. In petroleum engineering, its applicability covers a wide range including homogeneous and sectional homogeneous reservoir as well as the moving boundary problem [12, 13, and 14]. An earlier form of the BEM for a diffusivity equation employing a Laplace transformation (LT) had the main advantage of transforming the parabolic equation into an elliptic type which is readily amenable to the BEM [15]. Researchers have considered random heterogeneities as in the analysis of flow problems in heterogeneous media [15, 16].

Multiphase flow problem is still under development. It is well known that solutions of hyperbolic conservation law in terms of saturation equation can develop discontinuities, even for smooth initial data [17]. An alternative approach is to consider a modified version of the two-phase flow equations by adding small capillary pressure [18]. The tradeoff is that sharp saturation fronts might be artificially smeared due to the added viscosity. In addition, we apply pressure only formulation to avoid solving hyperbolic saturation equation directly.

2. Reservoir Models

In this section, we present the principle of conservation law, continuity and Darcy's law; briefly give the primary physical and geological parameters influencing fluid flow in porous media. With this we provide mathematical models for immiscible single-phase and multi-phase flow, assuming some common simplifications.

The conservation law in differential form is given as:

$$\partial(\rho\phi)/\partial t + \nabla\cdot(\rho v) = q \qquad (1)$$

where v is the volumetric velocity, ρ is the density, t is the time and q corresponds to source or sink terms. The continuity equation represents conservation of mass for fluid flow. For an incompressible fluid, the continuity equation can be derived from Eq. 1 as follows:

$$\Delta\cdot v = 0 \qquad (2)$$

Darcy's law describes the single phase flow of a fluid through a porous medium, and given as:

$$v = (-k/\mu)\nabla p \qquad (3)$$

where k is the permeability, μ is the viscosity and p is the pressure.

2.1 A Single Phase Homogenous Model

The mathematical expression for the unsteady state flow of a single-phase slightly compressible fluid in an isotropic porous media can be written as:

$$\nabla^2 p = (\phi c_t/k)(\partial p/\partial t) + (q\mu/\rho k) \qquad (4)$$

It is a common approach to work with dimensionless-quantities in order to keep values even if the scale or the properties of a well and reservoir are changed, or if the unit system changes. The following dimensionless variables are introduced:

$$p_D = (p_{ref} - p)/p_{ref},\ x_D = x/\sqrt{A}$$
$$y_D = y/\sqrt{A}, q_D = \mu q A/(k\rho p_{ref}) \qquad (5)$$
$$t_D = kt/(\phi\mu c_t A)$$

where A is the area, p_{ref} is the reference pressure. Then, Eq. 4 becomes:

$$\nabla^2 p_D = \partial p_D/\partial t_D + \sum_{l=1}^{n_w} q_{Dl}\delta(x_D - x_{Dl})\delta(y_D - y_{Dl}) \qquad (6)$$

where $\delta(z - z_0)$ is the Dirac-Delta function used to clarify the locations of n_w sinks and sources, and s is Laplace parameter. Applying Laplace Transformation to Eq. 5:

$$\nabla^2 \overline{p}_D = s\,\overline{p}_D - p_{Di} + (1/s)\sum_{l=1}^{n_w} q_{Dl}$$
$$\delta(x_D - x_{Dl})\delta(y_D - y_{Dl}) \qquad (7)$$

2.2 Two–Phase Immiscible Flow Model

Conservation must be posed for both phases. In addition, we have phase pressures that typically are related through a quantity called the capillary pressure. The capillary pressure reflects the fact that the pressures at each side of a fluid-fluid interface differ due to interfacial tension. The conservation of mass for each phase is:

$$\partial(\rho_i\phi s_i)/\partial t + \nabla\cdot(\rho_i v_i) = q_i \qquad (8)$$

where s_i is the saturation of phase i. Each phase may have distinct sources q_i and the volumetric flow velocity for each fluid is given by a generalized Darcy's law for multi-phase flow:

$$v_i = -K\lambda_i\nabla p_i \qquad (9)$$

where p_i is the pressure of fluid phase i. The relative mobility λ_i of phase i is defined by:

$$\lambda_i = k_{ri}/\mu_i \qquad (10)$$

A fully coupled model can be derived by substituting Eq. 9 and 10 into Eq. 8:

$$\partial(\phi s_w \rho_w)/\partial t - \nabla\cdot(\rho_w k_{rw}/\mu_w K \nabla p_w) - q_w = 0 \quad (11)$$

$$\partial(\phi s_n \rho_n)/\partial t - \nabla\cdot(\rho_n k_{rn}/\mu_n K \nabla p_n) - q_n = 0 \quad (12)$$

The following relations are needed:

$$s_w + s_n = 1 \quad (13)$$

$$p_n - p_w = p_c(s_w), \partial p_c/\partial p_w = -1 \quad (14)$$

$$\partial s_w/\partial t = \partial s_w/\partial p_c \times \partial p_c/\partial p_w \times \partial p_w/\partial t \quad (15)$$

If we assume fluid density, viscosity and absolute permeability are constant, substituting Eq. 13–15, and Eq. 11 becomes:

$$\begin{aligned}&(\phi \partial s_w/\partial p_c)(\partial p_w/\partial t) + k_{rw}/\mu_w K \nabla^2 p_w \\ &+ K/\mu_w \vec{\nabla} k_{rw} \vec{\nabla} p_w + q_w = 0\end{aligned} \quad (16)$$

Similar formulation can be applied to Eq. 16:

$$\begin{aligned}&(\phi \partial s_w/\partial p_c)(\partial p_n/\partial t) + k_{ro}/\mu_o K \nabla^2 p_n \\ &+ K/\mu_n \vec{\nabla} k_{ro} \vec{\nabla} p_n + q_n = 0\end{aligned} \quad (17)$$

In this model, gravity has been neglected.

3. Methodology

In this section, the BEM for the initial- and boundary-value problem presented in the previous section are set up. The expansion of the solution over all boundaries and the discretization over boundary surfaces and time are implemented.

3.1 The BEM Formulation for Single Phase

In a reservoir system with uniform initial pressure, we can obtain the solution of the 2D unsteady state flow problem in Laplace space from Eq. 7, which is associated with the modified Helmholtz $(\nabla^2 - s)$ operator:

$$\bar{G}(x_D, y_D; \zeta, \eta; s) = -1/2\pi K_0(r_D\sqrt{s}) \quad (18)$$

where r_D is the distance between the field point and the source point:

$$r_D = \sqrt{(x_D - \zeta)^2 + (y_D - \eta)^2} \quad (19)$$

Casting Eq. 7 into divergence form, integrating over the domain of the problem, using the shifting property of the Dirac delta function and the divergence theorem of Gauss, we obtain:

$$\begin{aligned}&\theta \bar{p}_D(x_D, y_D; s) = \int_\Gamma (\bar{G}(\partial \bar{p}_D/\partial n_D) \\ &- \bar{p}_D(\partial \bar{G}/\partial n_D))dS - \sum_{l=1}^{n_w} \bar{G}_l q_{Dl}/s\end{aligned} \quad (20)$$

Matrix equations are generated by substituting in Eq. 20 the space interpolation functions:

$$\sum_{i=1}^{N}(a_{ij}\bar{p}_{Dj} + b_{ij}\bar{p}_{Dnj}) = 0 \quad i = 1, N \quad (21)$$

The detailed derivation and expression of each term in Eq. 21 can be found in [15].

3.2 The DRBEM Formulation for Two-Phase Flow

In solving the Eq. 16 and 17 using the boundary element formulation, fundamental solution to the Laplace equation was applied, the domain integrals generated by the time and space derivatives are approximated using the dual reciprocity method, and then Green's theorem is applied to get a boundary solution. This procedure gives a boundary only solution with less complexity in the solution process and allows any of the coefficients to have variable values without adopting special techniques. Take Eq. 16 for the pressure in the wetting phase as an example, for simplicity, it can be expressed as

$$\nabla^2 p_w = b \quad (22)$$

where

$$\begin{aligned}&b = (-\phi\mu_w/Kk_{rw})(\partial s_w/\partial p_c)(\partial p_w/\partial t) \\ &-\nabla k_{rw}\nabla p_w/k_{rw}\end{aligned} \quad (23)$$

After the DRBEM formulation, Eq. 23 can be expressed in discretized form as:

$$\begin{aligned}&g_i p_{wi} + \sum_{k=1}^{N}\int_{\Gamma^k}(\partial p_w^*/\partial n)p_w d\Gamma - \sum_{k=1}^{N}\int_{\Gamma^k} p_w^*(\partial p_w/\partial n)d\Gamma = \\ &\sum_{j=1}^{N+L}\alpha_j(g_i\tilde{p}_{wij} + \sum_{k=1}^{N}\int_{\Gamma^k}(\partial p_w^*/\partial n)\tilde{p}_{wj}d\Gamma - \sum_{k=1}^{N}\int_{\Gamma^k} p_w^*(\partial \tilde{p}_{wj}/\partial n)d\Gamma\end{aligned} \quad (24)$$

where $g_i = \theta/2\pi$, θ is the internal angle between two boundary elements. N is the number of nodes on the boundary and L is the number of internal nodes which do not require any associated internal gridding. n is the unit vector normal to the boundary of the domain Ω. p_w^* is the fundamental solution of the Laplace equation and $\bar{p}_w$ is a particular solution.

3.3 Matrix Formulation and Assembly of Equations

After application of collocation technique to all boundary and internal nodes, Eq. 24 can be written in terms of four matrices which depend only on the geometry of the problem:

$$Hp_w - Gq_w = (H\tilde{p}_w - G\tilde{q}_w)\alpha \quad (25)$$

where $q_w = dp_w / dn$, and $\tilde{q}_w = d\tilde{p}_w / dn$, $\tilde{p}_w$ and $\tilde{q}_w$ are known once the approximation function f is defined. Also in the DRBEM:

$$b = F\alpha \quad (26)$$

The dual reciprocity method formulation for p_w is obtained after replacing the non-homogeneous term in Eq. 26 with Eq. 25:

$$\sum_{j=1}^{n} h_{ij} p_{wj} - \sum_{j=1}^{n} g_{ij} q_{wj} = \sum_{j=1}^{n} s_{ij} \{(-\phi\mu_w / K\tilde{k}_{rwj}) (\partial\tilde{s}_{wj} / \partial p_c)(\partial p_{wj} / \partial t) - (1/\tilde{k}_{rwj})[(\partial\tilde{k}_{rwj} / \partial x) (\partial p_{wj} / \partial x) + (\partial\tilde{k}_{rwj} / \partial y)(\partial p_{wj} / \partial y)]\} \quad (27)$$

where s_{ij} is the matrix $(H\tilde{p}_w - G\tilde{q}_w)F^{-1}$, $\tilde{k}_{rwj}$ and $\partial\tilde{s}_{wj} / \partial p_c$ are calculated using values of p_w, p_o from the previous iteration, which will be denoted by $\tilde{p}_w, \tilde{p}_o$ from here on.

$\partial p_{wj} / \partial x$ is obtained by applying the DRM approximation and similar for $\partial p_{wj} / \partial y$:

$$\partial p_{wj} / \partial x = \sum_{k=1}^{n} \sum_{l=1}^{n} (\partial f_{jl} / \partial x) f_{lk}^{-1} p_{wk} \quad (28)$$

The time discretization is based on the implicit/explicit Euler method:

$$p_w = (1-\theta_u) p_w^m + \theta_u p_w^{m+1} \quad (29)$$

$$q_w = (1-\theta_q) q_w^m + \theta_q q_w^{m+1} \quad (30)$$

The time derivative is approximated using a finite difference scheme:

$$\partial p_w / \partial t = (1/\Delta t)(p_w^{m+1} - p_w^m) \quad (31)$$

By applying Eq. 28 – 31, Eq. 27 can be recast as:

$$(\theta_u H + \tilde{R}_w / \Delta t + \theta_u \tilde{T}_w) p_w^{m+1} - \theta_q G q_w^{m+1} = \{(\theta_u - 1) (H + \tilde{T}_w) + \tilde{R}_w / \Delta t\} p_w^m + (1-\theta_q) G q_w^m \quad (32)$$

where $\tilde{R}_w$ is a matrix of components

$$\tilde{r}_{wij} = s_{ij} (\phi\mu_w / K\tilde{k}_{rwj})(\partial\tilde{s}_{wj} / \partial p_c) \quad (33)$$

$\tilde{T}_w$ is defined as

$$\tilde{T}_w = \tilde{D}_x (\partial F / \partial x) F^{-1} + \tilde{D}_y (\partial F / \partial y) F^{-1}$$

(34)

where $\tilde{D}_x$ has components:

$$d_{ij} = s_{ij} (1/\tilde{k}_{rwj})(\partial\tilde{k}_{rwj} / \partial x) \quad (35)$$

and similar for $\tilde{D}_y$.

In this approach, the formulated systems of governing equations are only partial pressure involved. Therefore, saturation can be directly calculated through a pre-set capillary pressure model.

3.4 Capillary Pressure Model

There are several functions that have been proposed to describe the relationship between the capillary pressure and saturation; among the most popular are those given by Leverett [18], Brooks and Corey [19] and van Genuchten [20]. The limitation of the VG model is that it does not consider the entry pressure which is especially important for heterogeneous porous medium. But it has the advantage that its derivatives are continuous, and usually expressed as:

$$p_c(s_w) = (1/\alpha)(s_{ew}{}^{1/m} - 1)^{1/n} \quad (40)$$

α , m and n are model parameters. The effective wetting saturation is given by:

$$s_{ew} = (s_w - s_{wr}) / (1 - s_{wr} - s_{or}) \quad (41)$$

The BC model is defined as:

$$p_c = p_d s_{ew}^{-1/\lambda} \quad (42)$$

3.5 Relative Permeability Model

The relative permeability curve is also a function of saturation and defined in van Genuchten model as:

$$k_{rw} = s_{ew}^{1/2}(1 - (1 - s_{ew}^{1/m})^m)^2 \quad (43)$$

$$k_{ro} = (1 - s_{ew})^{1/2}(1 - s_{ew}^{1/m})^{2m} \quad (44)$$

The BC model defines the relative permeability curve as the follows:

$$k_{rw} = s_w^{(2+3\lambda)/\lambda} \quad (45)$$

$$k_{ro} = (1 - s_w)^2(1 - s_w^{(2+\lambda)/\lambda}) \quad (46)$$

3.6 Well Treatment

Sources and sinks with diminishing radii are examples of singularities which can be addressed in two different ways. Being Dirac-delta functions for line source conditions, they only appear as additive in-homogeneous term in the flow equation. More precisely, this scheme defines the well as a source /sinks term. Another way to address the singularities is to use singularity programming. This approach lets us achieve the separation of well singularities and non-singular solutions. It was proven that singularity programming was a useful tool when applied in conjunction with the boundary element method [15, 16].

4. Numerical Results

4.1 Validation

Case 1: A vertical well at the center of a closed rectangular reservoir

The value of using singularity programming in combination with the BEM can be demonstrated by comparing simulated well test results to the available analytical solutions for a vertical well centered at a closed rectangular reservoir. We validate our model by simulating the pressure behavior of a vertical well in a process of drawdown.

Table 1 lists he reservoir rock and fluid properties used in this example. **Fig. 1** proves the excellent agreement of our BEM model and its analytical solution.

Table 1. Rock and fluid properties for case 1

Parameters	Value
Reservoir geometry	$900 \times 900 \times 15m^3$
Porosity	0.3
Viscosity	$0.001 Pa \cdot s$
Permeability	$1.E-10m^2$
Compressibility	$1.0E-2Pa^{-1}$
Wellbore radius	$0.2m$
Production rate	$1.5\ m^3/d$

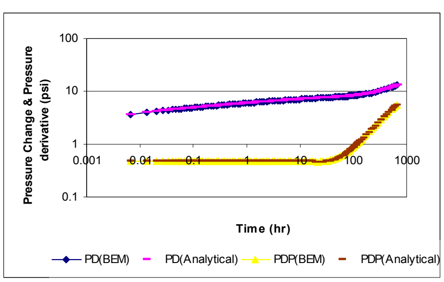

Figure 1. Drawdown pressure responses of case 1

In this section, we present an application of the BEM and the FASM method to multiphase flow in a porous medium. The reservoir simulation models developed in this study is applicable to various inner and outer boundary conditions, including no-flow boundaries and constant pressure boundaries.

Case 2: A mixed boundary problem

In this example, we construct a Cartesian grid of size $20 \times 20 \times 1$ cells to deal with the sink/source term instead of singularity programming. The left hand side of the reservoir boundary is Dirichlet type of $p = 10bar$ and the other sides are no–flow boundary. The fluid and rock properties are listed in **Table 2**.

Table 2. Rock and fluid properties for case 3

Parameters	Value
Reservoir geometry	$2000 \times 2000 \times 30m^3$
Porosity	0.3
Viscosity	$0.001 Pa \cdot s$
Permeability	$1.E-10m^2$
Wellbore radius	$0.1m$
Production rate	$2.5m^3/d$

We plot the computation results for pressure change and production performance in **Fig. 2–3**. The results show the flexibility of the BEM to varied boundary types as well as multiple approaches to the sink/source terms.

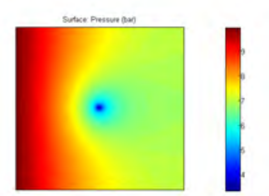

Figure 2. Pressure profile of case 2 by BEM

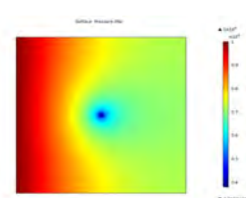

Figure 3. Pressure profile of case 2 by Comsol

4.2 Application

Case 3: Inverted five-spot waterflooding

We confine our attention to a two–dimensional, two–phase waterflooding problem through the DRBEM approach verified in the previous section. It is a quarter of inverted five-spot pattern where water is injected in an oil reservoir from the lower left and oil is produced from the upper right as shown in **Fig. 4**. We express the physical problem by taking the following values of the parameters from **Table 3**.

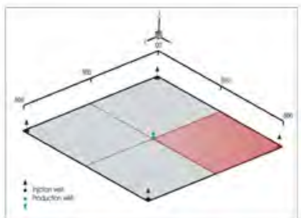

Figure 4. Reservoir geometry and discretization scheme for case 3

Table 3. Reservoir rock and fluid property for case 3

Symbol and Meaning	Value	Unit
domain, Ω	$(0, 300)^2$	m
oil viscosity, μ_o	0.01	Pa s
water viscosity, μ_w	0.001	Pa s
permeability (vertical) , k_v	1.0e-011	m^2
permeability (horizontal) , k_h	1.0e-011	m^2
porosity, n	20	%
model	*Brooks-Corey*	
BC-parameter, λ	2	-
entry pressure, p_d	10e04	Pa
residual saturation of oil, s_{ro}	0.15	-
residual saturation of water, s_{rw}	0.2	-
boundary condition		
outlet pressure, p_{out}	10e05	Pa
the average flux, $(u.n_e)_{in}$	h^{-1}*1.05e-04	m/s
Initial condition		
water saturation, s_w	$s_{w0}=s_{rw}$	

The boundaries of the domain are taken as no–flow boundaries. Injector operates under rate control, and producer is under pressure control. The reservoir is filled with oil initially, and the pressure variable does not require an initial condition due to the elliptic nature of the equation. The boundary is discretized into a fairly coarse of 160 elements, and the domain is divided into 160 sub-regions to increase the stability and accuracy which may be affected by the traditional single domain DRBEM in case of large velocity. In **Fig. 5** we show the pressure profile computed for both oil and water phase at the injector. And **Fig. 6** shows the rate profile calculated at the producer. **Fig. 7** gives the saturation profile at time t=100 and 300days computed from our DRBEM algorithm and **Fig. 8** is the simulation result from Comsol.

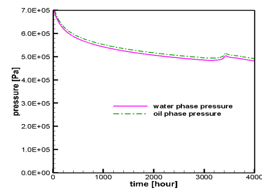

Figure 5. Injector Oil and water pressure profile

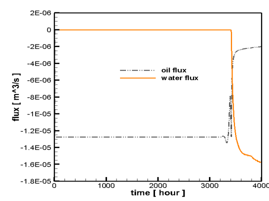

Figure 6. One of the four producers production profile

Figure 7. Water saturation profile at time t=100, and 300 days

Figure 8. Water saturation profile at time t=100, and 300 days by Comsol

5. Conclusions

The analytical based numerical models developed in this study can accurately predict performance of vertical wells under single phase and oil-water two-phase flow conditions. Firstly, the advantage of LT is that time appears as a parameter in the Laplace formula which reduce the computation cost by convolution. The disadvantage is that the kernel (Green's) function becomes more complicated and the majority of the integrals arising from the boundary discretization must be taken care of numerically. Secondly, the BEM in conjunction with singularity programming increased the accuracy of the more sensitive pressure derivative result. Thirdly, for the transport equation, the DRBEM was combined successfully with method of fundamental solution for convective terms. Fourthly, more accurate result can be achieved with higher density of internal nodes. Finally, separating the problem domain into several sub-regions can increase the stability and accuracy in case of large velocity.

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