Application of a Weakly Non-Linear Analysis to the Analysis of Thermoacoustic Combustion Instabilities

D. Laera[1], S.M. Camporeale[1*], G. Campa[2],
[1]Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management (DMMM), Via Re David, 200 Bari, Italy
[2]Ansaldo Energia, Via Nicola Lorenzi, 8, 16152, Genova, Italy
*Corresponding author: sergio.camporeale@poliba.it

Abstract: The thermoacoustic combustion instabilities are complex phenomena that may occur in steady flow combustion systems that are aboard of, e.g., rocket engines or gas turbines. The phenomena involve the interaction of chemical reactions with fluid-dynamic and propagation of pressure waves in the combustion chamber. In some cases, combustion instabilities may lead to a stable condition (known as "limit-cycle") characterized by pressure oscillations so intense that the operation of the engine cannot be continued.

Over the years, it has been found that a numerical method suitable to study this phenomenon is to model the propagation of the linearized ("acoustic") pressure oscillations by means of inhomogeneous wave equation in the frequency domain and considering the heat release fluctuations (q') produced by the flame dynamics as source term.

This simulation can be carried out in COMSOL Multiphysics through the Pressure Acoustic Module (acpr).

A Flame Transfer Function (FTF) is usually used to model heat release fluctuations as a delayed function. Solving the eigenvalue problem, a linear stability analysis is performed. By means of this methodology, it is possible to estimate frequency, wave shape and growth rate of the resonant modes. However, a nonlinear analysis must be applied in order to estimate the amplitudes of limit cycles. To this purpose, the influence of a Non-Linear Flame Describing Function (NLFDF) function is examined. Considering that the nonlinear terms can be considered relatively small with respect to the linear ones, a weakly non-linear analysis can be performed to identify the process that characterizes the onset and the growth oscillations.

In the present paper the methodology adopted in COMSOL to simulate the non-linear dynamics is shown and the results of the application to a simple cylindrical combustor are reported.

Keywords: Thermoacoustic, weakly non-linear analysis, pressure acoustic module,

1. Introduction

The mechanisms leading to the onset of instability can be grouped into two categories: linear and nonlinear. In a linearly unstable system, a small perturbation determines the instability. This kind of system is generally not observed in nature. On the other hand, a nonlinearly unstable system can be stable to small perturbations but it can become unstable if the initial disturbance is larger than some threshold. This behavior is known as '*triggering*' (Juniper, 2011).

A crucial step in the understanding of the thermoacoustic combustion instabilities is the model of the heat release fluctuations. These instabilities are due to coupling mechanism between the unsteady heat release rate and the pressure oscillations inside the combustor. In order to study these instabilities, heat release fluctuations are usually coupled to velocity fluctuations through linear correlations. Linear flame models are able to predict whether the non-oscillating steady state of a thermoacoustic system is "asymptotically" stable (without oscillations) or unstable (increasing oscillations). However, linear models describing the system behavior are not able to predict triggering instabilities and limit cycles. In order to get this kind of information, non linearities must be introduced into the model and the analysis.

The behavior of nonlinear systems can be reported in relation to the variation of a control parameter. The changes in the system dynamics are called bifurcations and the parameter values at which they occur are called bifurcation points (Strogatz, 1994).

In the years, several techniques have been proposed in order to track the bifurcation diagrams. In one of them, a systematic variation of parameters is carried out and the behavior of the system is examined by direct time integration (Mariappan & Sujith, 2010). However this method is computationally expensive.

Another method for obtaining the bifurcation diagrams is the so-called numerical continuation (Jahnke & Culick, 1994). This approach is based on the iterative solution of a set of parameterized nonlinear equations given an initial guess. The diagram is tracked varying a parameter and including the solutions which satisfy the set of equations for a given state of the system. The unstable limit cycle can also be computed. Compared to other methods, it is very efficient in obtaining the dependence of the solution from the control parameter. However, it takes a long time to map the bifurcation diagram and it can be also too computationally expensive. Thanks to improvements in the method and in the parallel computing, continuation methods are likely to become important tools in nonlinear analysis of thermoacoustic (Juniper M. , 2012).

The use of low-order network models to map the bifurcation diagram as a function of a control parameter has been shown by Campa and Juniper (Campa & Juniper, 2012). Instead of numerically integrate the fully non-linear equations governing the phenomenon, the authors proposed a weakly nonlinear approach consisting in a linear eigenvalue analysis around a non-linear steady state of the system. This approach, less computationally expensive than continuation method will be used in this work

The use of a framework based on the finite element method (FEM) to study nonlinear flame models was proposed by Pankiewitz and Sattelmayer (Pankiewitz & Sattelmayer, 2003), who examined in the time domain a three-dimensional combustion chamber, predicting the amplitude of limit cycles determined by a nonlinear flame model.

Within a similar FEM framework, the feasibility of mapping the bifurcation diagrams in the frequency domain is shown in this paper. The bases of this framework are described in previous works of ours considering linear flame models (Camporeale, Fortunato, & Campa, 2011). Making use of the Pressure Acoustic Physics Interface (acpr) of the Acoustics Module of COMSOL Multiphysics (Campa & Camporeale, A novel FEM method for predicting thermoacoustic combustion instability, 2009), the approach numerically solves the differential equation problem converted in a complex eigenvalue problem in the frequency domain. The eigenvalue problem is solved by means of a linearization under the hypothesis of small oscillations. From the complex eigenvalues of the system it is possible to ascertain if the corresponding mode is unstable or if the oscillations will decrease in time, i.e. the mode is stable (Camporeale, Fortunato, & Campa, 2011). Through the proposed approach it is possible to perform a weakly nonlinear study and to map bifurcation diagrams not only for simple configurations, but also for industrial complex configurations.

In the first section of the paper the bases of the nonlinear analysis are explained. In the second section the FEM approach is described, explaining the procedure adopted to track the bifurcation diagrams using COMSOL Multiphysics. In the third section, the results are shown, analyzing the bifurcations occurring in a simple Rijke tube for two di different nonlinear flames.

2. Nonlinear Analysis

Figure 1 shows two different types of bifurcation behaviors that the system can undergo when a control parameter R varies. The variable on the y-axis is the limit cycle amplitude. At low values of R, the amplitude is zero that means that the system is stable and no combustion dynamics are registered. (solid line in Fig. 1). When R reaches the Hopf bifurcation point, the system becomes unstable in the sense that, increasing the value of the control parameter, the condition at zero amplitude becomes unstable (dashed line in Fig. 1) and the system start to oscillate with increasing amplitude until the limit cycle is reached (solid line at non-zero amplitude). In the supercritical bifurcation (Fig. 1(a)), with increasing R beyond the Hopf point (see red arrows in Fig. 1(a)), a gradually increase of the limit cycle amplitude is obtained. In the subcritical bifurcation (Fig. 1(b)), while gradually increasing the parameter R, there is a jump from zero to large amplitude when the Hopf point is crossed (see red arrows in Fig. 1(b)).

In the supercritical bifurcation, while decreasing R, the limit cycle amplitude is gradually reduced when R approaches the Hopf point and, then, returns to zero. In the subcritical bifurcation, instead, once the limit cycle is established, even for values of R lower than the one corresponding to the Hopf point, the amplitude of the limit cycle continue to follow the large amplitude branch, until the fold point is reached. Further decreasing R, there is a jump to zero as shown by the blue arrow path in Fig. 1(b). The dashed line at non-zero amplitudes in Fig. 1(b) is the locus of the unstable equilibrium point of the system (Strogatz, 1994). Then, in a thermoacoustic system characterized by a subcritical bifurcation, even when it is in a linearly stable condition, a limit cycle condition may be "triggered" by a sufficiently large impulse (Juniper M. , 2012).

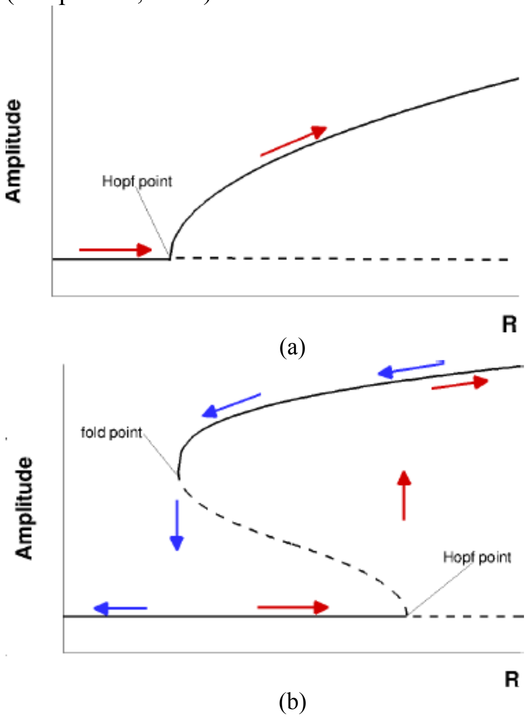


Figure 1. Steady state oscillation amplitude as a function of R for (a) a supercritical bifurcation and (b) a subcritical bifurcation (Campa & Juniper, 2012). As the control parameter R is increased, the system follows the red arrow path. As it is decreased, the system follows the blue arrow path.

3. Finite Element Method Approach

The nonlinear analysis is carried out by using the Pressure Acoustic Physics Interface (acpr) of the Acoustics Module of COMSOL Multiphysics. The code, based on a Finite Element Method (FEM) is able to analyze three-dimensional geometries. This approach numerically solves the differential equation problem converted in a complex eigenvalue problem in the frequency domain and the stability analysis can be conducted.

The fluid is regarded as an ideal gas. The effects of viscosity, thermal diffusivity and heat transfer are neglected; the mean pressure is assumed uniform in the domain. The mean flow velocity $\bar{u}$ is much lower than the speed of sound, so its influence on the propagation of the pressure waves has been neglected. Under such hypotheses, in presence of heat fluctuations, the inhomogeneous wave equation can be obtained (Dowling & Stow, Acoustic analysis of gas turbine combustors, 2003)

$$\frac{1}{\bar{c}^2}\frac{\partial^2 p'}{\partial t^2} - \bar{\rho}\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla p'\right) = \frac{\gamma-1}{\bar{c}^2}\frac{\partial q'}{\partial t} \qquad (1)$$

where p' is the pressure fluctuation, q' is the heat release fluctuation, γ is the ratio of specific heats, ρ is the density and c is the speed of sound. Under the assumption of zero mean flow velocity, neglecting the effects of the temperature variation, no entropy waves are considered and the pressure fluctuations are related to the velocity fluctuations by

$$\frac{\partial \mathbf{u}'}{\partial t} + \frac{1}{\bar{\rho}}\nabla p' = 0 \qquad (2)$$

Heat release fluctuations q' are coupled to the velocity fluctuations u' taken at the injection point upstream the flame zone with a time delay τ. In the linear case and in the time domain it means

$$\frac{q'}{\bar{q}} = -\kappa\frac{u_i'(t-\tau)}{\bar{u_i}} \qquad (3)$$

where subscript i refers to the injection point and κ is the intensity index, which represents a dimensionless parameter of proportionality between the heat release fluctuations and the velocity fluctuations.

For the search of the eigenvalues and the eigenmodes of the system, the analysis is performed in the frequency domain and the fluctuating variables are expressed by complex functions of time and position with a sinusoidal form: $p' = \hat{p}\exp(i\omega t)$, where $\omega = \omega_r + i\omega_i$ the complex frequency. Its real part ω_r gives the frequency of oscillations, while the imaginary part ω_i provides the growth rate at which the amplitude of the oscillations increases per cycle. The linear flame model in the frequency domain, assuming a harmonic form of the fluctuation variables, can be written as

$$\frac{\hat{q}}{\bar{q}} = -\kappa\frac{\hat{u_i}}{\bar{u_i}}\exp(-i\omega\tau) \qquad (4)$$

Then, starting from Eq. (1) the Helmholtz equation in the frequency domain governing the acoustic pressure waves can be written

$$\frac{\lambda^2}{\bar{c}^2}\hat{p} - \bar{\rho}\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla\hat{p}\right) = -\frac{\gamma-1}{\bar{c}^2}\lambda\hat{q} \qquad (5)$$

where $\lambda = -i\omega$ is the eigenvalue.

3.1 Nonlinear Flame Model

Nonlinearities are introduced in the flame model. An evolution of the usual κ-τ model is used (Crocco & Zheng, 1956). For the nonlinear flame model, the flame transfer function can be obtained following the procedure proposed by Dowling (Dowling, A kinematic model of ducted flame, 1999) and recalled in (Campa & Juniper, Obtaining bifurcation diagrams with a thermoacoustic network model, 2012). The nonlinear flame transfer function can be expressed as multiple of the linear flame transfer function and a function of frequency and oscillations amplitude $r = |\hat{u}/\bar{u}|$

$$T_{flame}^{NL}(\omega, r) = T_{flame}^{L}(\omega, r) \cdot NFTF(r) \quad (6)$$

where the (linear) flame transfer function is defined by

$$T_{flame}^{L}(\omega) = \frac{\hat{q}/\bar{q}}{\hat{q_\iota}/\bar{q_\iota}} = -\kappa \exp(-i\omega\tau) \quad (7)$$

and NFTF is the function deriving from the nonlinearity introduced into the flame model (Campa & Juniper, Obtaining bifurcation diagrams with a thermoacoustic network model, 2012).

The appropriate analysis for determining the nature of a Hopf point is a weakly nonlinear analysis. The procedure is a continuation method similar to the one described by Jahnke and Culick (Jahnke & Culick, 1994). The general technique is based on fixing all parameters of the system but one and tracing the steady states of the system as a function of this parameter. In this work, the control parameter of the bifurcation diagram is the intensity index κ, while the only not fixed parameter is the amplitude r of the limit cycle oscillations. It implies that, each time a value of the amplitude r is assumed to detect the solutions of the eigenvalue problem, a linear model is solved. In fact, if r is fixed, the NFTF function degenerates to a constant value and the transfer matrix become linear. As a consequence, although the flame model is nonlinear, the eigenvalue problem is solved for a linear flame model detecting each point of bifurcation diagram.

Firstly, the value of κ corresponding to the Hopf bifurcation point, κ_{Hopf}, is searched for, setting to zero the amplitude r, so that the flame model becomes linear. Then, for values of κ higher than κ_{Hopf}, the *regula falsi* method is adopted to detect the point corresponding to zero growth rate, i.e., a limit cycle condition. The entire procedure has been automatized using MATLAB® scripts integrated in COMSOL Multiphysics with *LiveLink™ for MATLAB®*.

4. Application

The previously described procedure is used to obtain the bifurcation diagrams of a simple Rijke tube assuming two different laws for the nonlinear flame transfer function: a third-order and a five-order polynomial law.

Figure 2 shows the computational domain and the mesh used for simulations. Open-end inlet and outlet boundary conditions, $p' = 0$, are considered. The dominion is subdivided in two parts representing the plenum and the combustion chamber; the location of the heat release zone is highlighted in blue. A uniform temperature of 300 K and of 700K is assumed for plenum and combustion chamber, respectively. The temperature increases linearly from 300 K to 700 K across the combustion zone. The heat release fluctuations q' are coupled to the velocity fluctuations u' with a time delay τ, which is assumed to be constant.

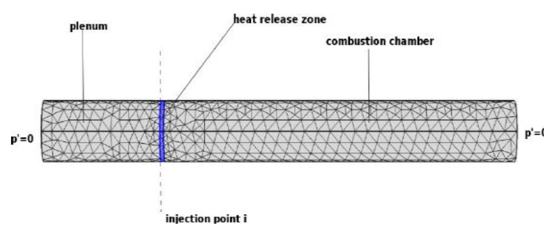


Figure 2. Computational mesh of the simple Rijke tube.

4.1 Damping

All the mechanisms of damping of acoustic energy can be grouped into three categories: convection through domain boundaries, thermal and viscous dissipation, energy transfer to higher modes. Due to the demonstrative nature of this application, only the first two damping mechanisms are discussed here.

In order to evaluate the acoustic energy losses due to the boundary condition, let's consider the conservation equation of the acoustic energy (Poinsot & Veynante, 2001)

$$\frac{\partial e_1}{\partial t} + \nabla \cdot f_1 = 0 \quad (8)$$

where $e_1$is the acoustic energy and $f_1$is the flux defined as $f_1 = p\overrightarrow{u_1}$. Eq. (8) shows that the changes in the total energy are only due to fluxes crossing the boundaries. For the system examined an open end boundary condition is set at the ends of the tube ($p' = 0$). So, the flux of acoustic energy is null at both boundaries which means that the system cannot dissipate acoustic energy through the boundaries. The acoustic energy initially present in the computational domain is constant.

Viscous and thermal losses through boundary layers are modelled as the first derivative of p' multiplied by a suitable damping coefficient ζ, following the procedure described in (Munjal, 1987). Introducing this term, the one-dimensional inhomogeneous damped wave equation is derived

$$\frac{1}{\bar{c}^2}\frac{\partial^2 p'}{\partial t^2} + \frac{\zeta}{cR}\frac{\partial p'}{\partial t} - \bar{\rho}\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla p'\right) = \frac{\gamma - 1}{\bar{c}^2}\frac{\partial q'}{\partial t} \quad (9)$$

where R is the radius of the duct. The damping coefficient for j-th mode is modelled as $\zeta_j = c_1 j^2 + c_2 j^{1/2}$ where c_1 and c_2 are constant for each mode. In this study, the maximum values assumed are 0.05 and 0.01, respectively. More information about the evaluation of these coefficient can be found in (Matveev, 2003).

4.2 First nonlinear flame model

The first nonlinear flame model considered in this work is a third-order polynomial law

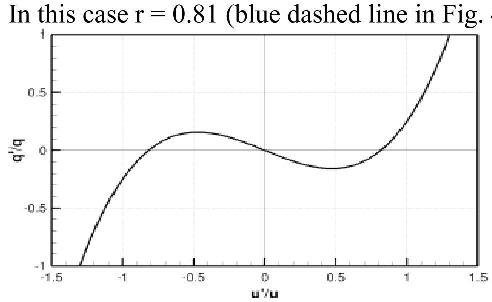
$$\frac{q'(t)}{\bar{q}} = -\kappa\left[\mu_2\left(\frac{u'(t-\tau)}{\bar{u}}\right)^3 + \mu_0\frac{u'(t-\tau)}{\bar{u}}\right] \quad (10)$$

where the coefficients $\mu_2$and μ_0 are equal to -1 and 0.5, respectively. The function NFTF for this model results

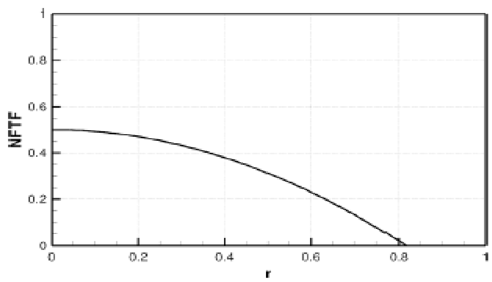
$$NFTF = \frac{3}{4}\mu_2 r^2 + \mu_0. \quad (11)$$

The pattern of the nonlinear flame model in Eq. (10) is shown in Fig. 3(a), whereas the NFTF is shown in Fig. 3(b). The NFTF is considered only when is positive and for positive values of the amplitude r to ensure the physical meaning of the flame model.

A supercritical bifurcation behavior of the system can be observed in Fig. 4. For both values of delay time analyzed, when the damping is neglected (ζ=0), the system is unstable for any value of the control parameter κ. Furthermore, the amplitude of the limit cycle is constant and corresponds to the value that saturates the NFTF. In this case r = 0.81 (blue dashed line in Fig. 4).



(a)



(b)

Figure 3. (a) Flame model (tracked for κ = 0.25) and (b) NFTF function

The bifurcation diagram is referred to the first axial mode and tracked for three different values of the damping coefficient (ζ), ζ = 0; ζ = 0:05; ζ = 0:09, assuming two different values of the time delay τ (8 and 11 ms). In particular, the value of 11 ms of τ is the delay time at which the system shows the maximum growth rate with linear FTF.

When considering ζ non-zero, the position of the Hopf point moves towards higher values of the control parameter κ as the damping level increases. With τ= 11 ms, (solid lines in Fig. 4), the bifurcation occurs at κ_H= 0.12 with ζ = 0.03 and κ_H= 0.33 with ζ = 0.09. Increasing ζ, an increase of the energy dissipation rate causes a reduction of the amplitude of oscillations. Due to saturation of the heat release, regardless of the damping level, at higher values of κ the amplitude of the stable limit cycle solution tends asymptotic to the value which nullify the NFTF. This result is highlighted in the zoom frame of Fig. 4. The influence of the time delay is qualitatively the same at all the damping levels: for a fixed value of κ, with a time delay of 8 ms, a lower limit cycle amplitude is registered (dot–dashed lines). Quantitatively this behavior is not the same for different value of damping; the amplitude reduction is greater for systems with higher damping (dot–dashed lines vs continuous lines).

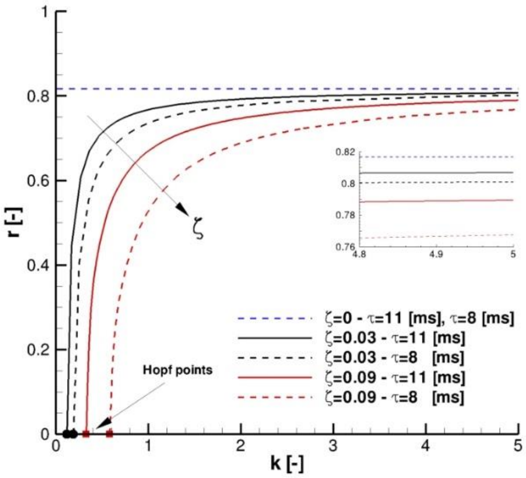


Figure 4. Bifurcation diagram for three value of damping coefficient ζ assuming two values of time delay τ

4.2 Second nonlinear flame model

A fifth-order polynomial law is assumed as nonlinear flame model

$$\frac{q'(t)}{\bar{q}} = -\kappa\left[\mu_4\left(\frac{u'(t-\tau)}{\bar{u}}\right)^5 + \mu_2\left(\frac{(t-\tau)}{\bar{u}}\right)^3 + \mu_0\frac{u'(t-\tau)}{\bar{u}}\right] \qquad (12)$$

where μ_4, μ_2 and μ_0 are coefficients equal to -1, 1 and 0.2, respectively. The function NFTF for this model results to be

$$NFTF = \frac{5}{8}\mu_4 r^4 + \frac{3}{4}\mu_2 r^2 + \mu_0 \qquad (13)$$

The pattern of the nonlinear flame model in Eq. (12) is shown in Fig. 5(a), whereas in Fig. 5(b) the NFTF is shown.

The bifurcation diagram for this case is shown in Fig. 6. A subcritical bifurcation is registered in this case. A solid line is used for stable points, while a dashed line is used for unstable points. The influence of the time τ on the bifurcation diagrams is investigated for only one value of damping coefficient ζ=0.09 and two different values of τ: 11 and 9 ms.

As registered for the previous case, the influence of τ is on the position of the Hopf point. The fold point seems not be influenced by the time delay. Again, at high values of κ, the influence of the damping coefficient on the amplitude of the limit cycle solution tends to become very small (differences highlighted in the zoom frame).

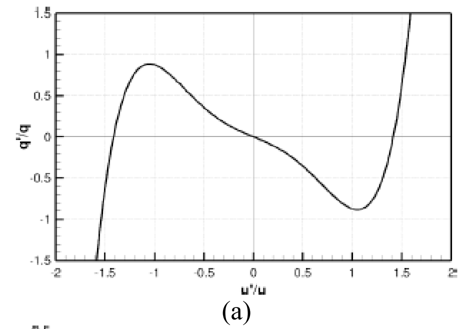


Figure 5. (a) Flame model (tracked for κ = 0.25) and (b) NFTF function

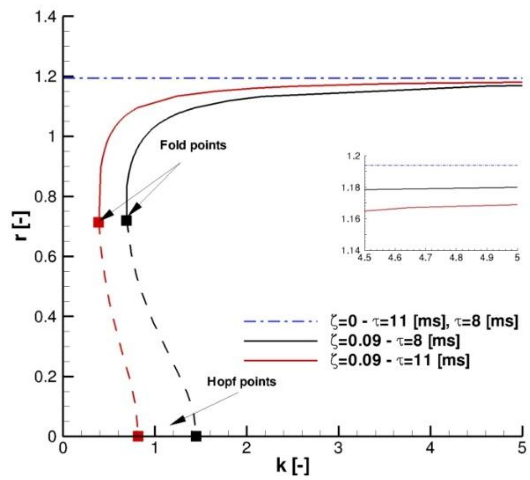


Figure 6. Bifurcation diagram for two different value of time delay for the nonlinear flame model

4. Conclusions

A weak nonlinear analysis has been performed to study the nonlinear behavior of a simple longitudinal combustion system. The nonlinear behavior of the system is considered within the flame model and a damping model. Heat release fluctuations are coupled to the velocity fluctuations through a nonlinear polynomial correlation. The behavior of the system is determined by the nature of the Hopf bifurcation.

The Pressure Acoustic Physics Interface (acpr) of the Acoustics Module of COMSOL Multiphysics is used to track the bifurcation diagrams.

The kind of bifurcation depends on the nonlinear flame model, independently of the geometrical configuration examined, following the predictions of the weakly nonlinear analysis. The influence of the time delay and the damping coefficient is investigated. The main points are:

1. the amplitude of the limit cycle solutions and bifurcation point is greatly influenced by the rate of acoustic energy dissipated;

2. due to saturation of the heat release, regardless of the damping level, at high values of κ, the amplitude of the stable limit cycle solution tends asymptotic to the value which nullify the NFTF;

3. time delay has a significant influence on the Hopf point. The minimum value of κ at which the bifurcation occurs is obtained with the value of the time delay that causes the resonant mode with the greatest growth rate.

Considering what has been said, the proposed approach proves to be able to treat nonlinear problems and to be applied to complex geometries in an industrial environment. Numerical and experimental data can be introduced into the shown framework, performing parametric analyses which can be helpful both in the design and in the check stage of a burner.

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