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Numerical solution of nonlinear PDEs exhibiting soft bifurcations

Alexei Boubitch
Dr. habil.,
Researcher,
IEE Luxembourg

Abstract

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- This talk communicates a numerical pseudo-dynamic approach to solve nonlinear stationary equations exhibiting bifurcations. The solution is achieved by passing from the stationary partial differential equation - to a pseudo-time dependent one. The latter is constructed in such a way that the desired nontrivial solution of the stationary equation represents its fixed point. The numeric solution of the stationary equation which is being looked for is then obtained as the solution of the pseudo-time dependent equation at a high enough value of the pseudo-time.

A nonlinear PDE in a most general form

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$$\boldsymbol{F}[\boldsymbol{u}(\boldsymbol{x}), \lambda] = 0$$

$$\boldsymbol{u}(\boldsymbol{x})|_{\partial\Omega} = 0$$

$$\boldsymbol{F}(0, \lambda) \equiv 0$$

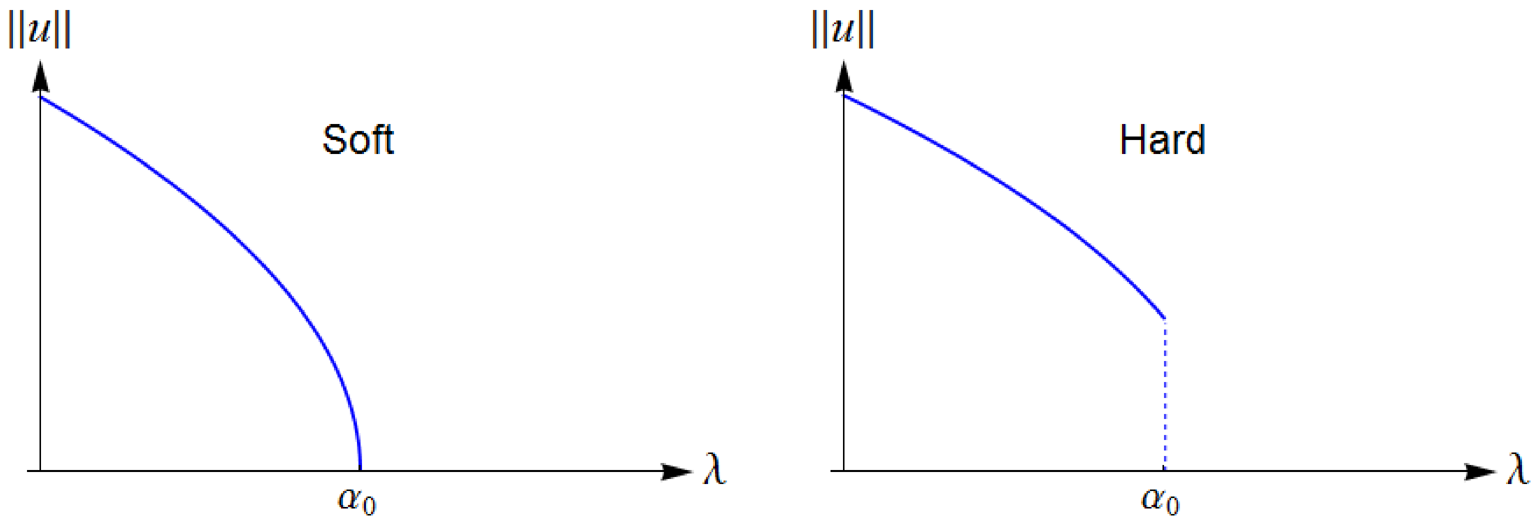
- A bifurcation
- At $\lambda > \alpha_0$

  Only a trivial solution $u(x) \equiv 0$ of the equation $F[u(x), \lambda] = 0$ exists
- At $\lambda \leq \alpha_0$

  One or several nontrivial solutions $u(x) \neq 0$ of the equation $F[u(x), \lambda] = 0$ emerge.

A soft and hard bifurcation: just to recall

A soft is a bifurcation taking place continuously in the bifurcation point. A hard bifurcation takes place jump-wise.



Here I only address soft bifurcations. Hard ones can be briefly discussed in the end.

A pseudo-dynamic approach

- Instead of the stationary equation F(u,λ)=0 let us study the behavior of the pseudo-time dependent equation.

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},\, t)}{\partial t} = \boldsymbol{F}[\, \boldsymbol{u}(\boldsymbol{x},\, t),\, \lambda] \quad \Longleftrightarrow \quad \boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) \rightarrow \boldsymbol{u}_S(x), \quad t \rightarrow \infty$$

$$\boldsymbol{u}(x,t)|_{\partial\Omega} = 0$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \not\equiv 0$$

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$F(u,\lambda)$, or $-F(u,\lambda)$, that's the question...

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{F}[\,\boldsymbol{u}(\boldsymbol{x},t),\lambda]$$

or

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{F}[\,\boldsymbol{u}(\boldsymbol{x},t),\lambda]$$

In one of these two cases the solution converges

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) \rightarrow \boldsymbol{u}_S(x), \quad t \rightarrow \infty$$

in the other it diverges

$$\boldsymbol{u}(\boldsymbol{x},t) \rightarrow \infty, \quad t \rightarrow \infty$$

Example: a 1D Ginzburg-Landau equation

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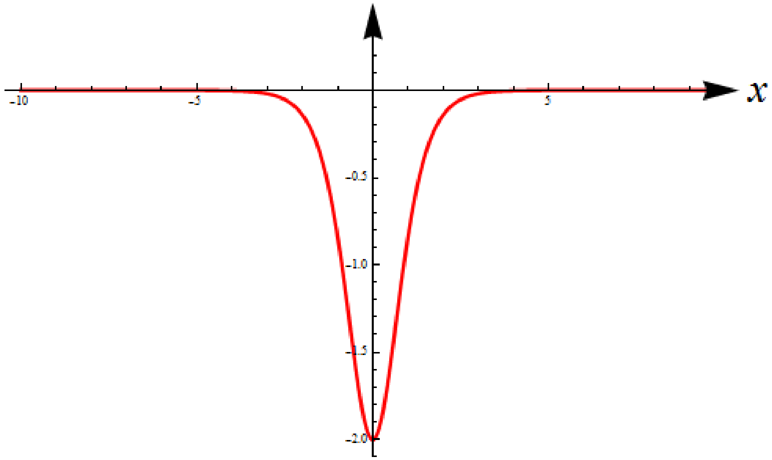
$$\frac{\partial^2 u(x)}{\partial x^2} = 2[\lambda + U(x)]\, u(x) + u^3(x)$$

$$u(\pm\infty) = 0$$

$$U(x) = -\frac{1}{\cosh^2(x)}$$



What does the bifurcation theory tell us?

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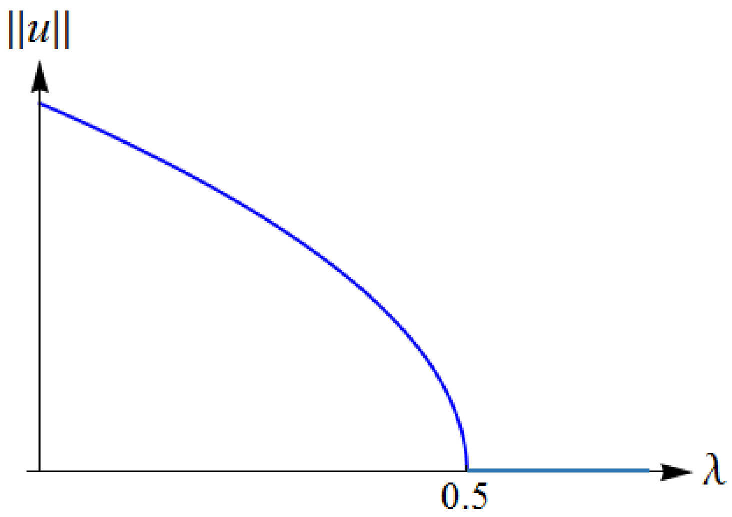
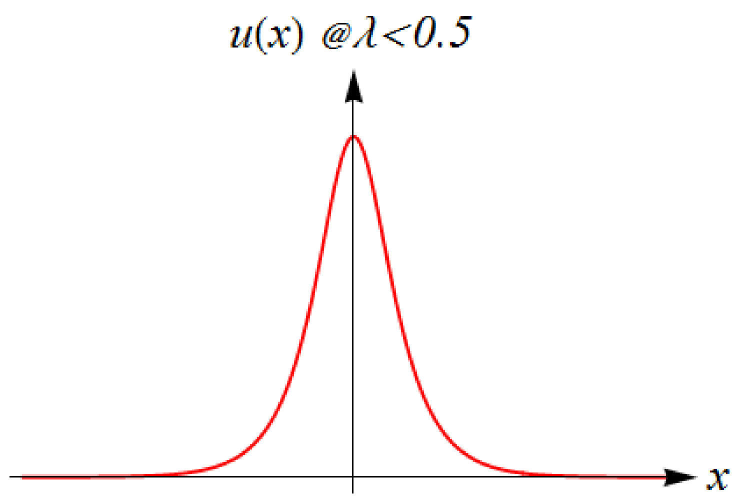
The bifurcation point: $\alpha_0 = \frac{1}{2}$

The branching off solution: $u(x) = \begin{cases} 0, & \lambda > 1/2 \\ \pm\sqrt{3}\,(1/2 - \lambda)^{1/2} \times \cosh^{-1}(x), & \lambda \le 1/2 \end{cases}$





Let us first solve it directly

- Switch to the COMSOL at the COMSOL model with the stationary solution

Pseudo-time dependent 1D Ginzburg-Landau equation...

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$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - 2\left[\lambda - \frac{1}{cosh^2(x)}\right] u(x,t) - u^3(x,t)$$

$$u(\pm L\ t) = 0$$

$$u(x,0) = u_0(x) \neq 0$$

Let is solve it from *t=0* to *t=T,*
the maximum pseudo-time of simulation.

… and its numerical solution

Let us now look at the COMSOL model with the pseudo-time dependent solution

Solution norm and the convergence control

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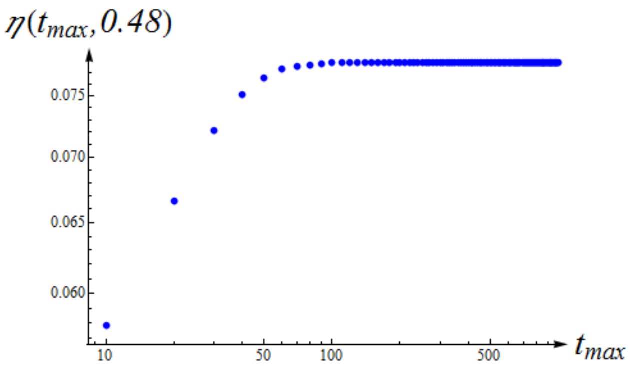
- The proximity of the solution to the fixed point can be estimated using the Hilbert norm:

$$\eta^2(t,\lambda) = \frac{1}{2L}\int_{-L}^{L} u^2(x,t)\,dx$$

Let us have a look at the COMSOL model with the pseudo-time dependent solution



$$T = \frac{T_0}{|\lambda - \alpha_0|}$$

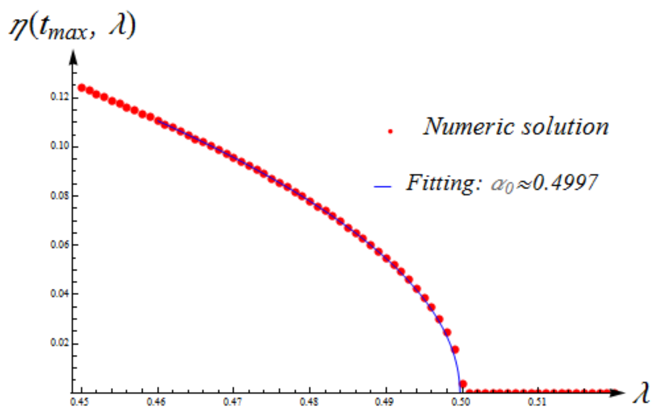
In search of the bifurcation point

According to the general theory one expects the behavior:

$$\eta(\lambda) = \begin{cases} 0, & \lambda > \alpha_0 \\ \eta_0 (\alpha_0 - \lambda)^\beta, & \lambda \le \alpha_0 \end{cases}$$

Here η_0, α_0 and β are fitting parameters.

Let us now switch to the COMSOL model where the dependence of the norm, $||u||$ on λ is determined



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Summary

- I shared an approach enabling one to solve numerically nonlinear stationary partial differential equations exhibiting a bifurcation.
- The approach is based on solving a pseudo-time dependent equation instead of the stationary one.
- The penalty is, however, an increased simulation time.
- Exibifurcation to take place. It can be applied to any nonlinear equation.

Take home: How to operate?

- Step 1: Go from $F(u,\lambda)=0$ to $\partial u/\partial t=F(u,\lambda)$
- Step 2: Solve it with the standard routine
- Step 3: Use an appropriate norm to control convergence, and find a good T
- Step 4: Keep in mind the that in the vicinity of bifurcation point (if any) the critical slowing down occurs.